Basic Queuing Systems

- What is queuing theory?
  - Queuing theory is the study of queues (sometimes called waiting lines)
  - Can be used to describe real world queues, or more abstract queues, found in many branches of computer science, such as operating systems
- Basic queuing theory

  Queuing theory is divided into 3 main sections:
  - Traffic flow
  - Scheduling
  - Facility design and employee allocation

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Kendall’s Notation

- **D.G. Kendall in 1951 proposed a standard notation for classifying queuing systems into different types.**
- **Accordingly the systems were described by the notation A/B/C/D/E where:**

A	**Distribution of inter arrival times of customers**
B	**Distribution of service times**
C	**Number of servers**
D	**Maximum number of customers in the system**
E	**Calling population size**

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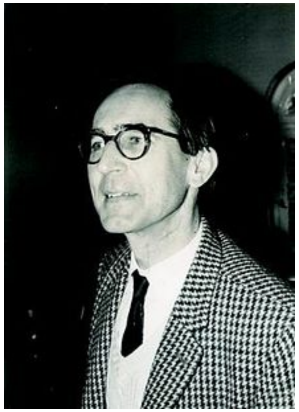
Kendall's Notation

A	顧客到訪間隔間之分佈
B	服務時間之分佈
C	伺服器數量
D	系統最大顧客數
E	通話人口大小



David George Kendall

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Kendall's Notation

A and B can take any of the following distributions types:

M	Exponential distribution (Markovian)
D	Degenerate (or deterministic) distribution
E_k	Erlang distribution (k = shape parameter)
H_k	Hyper exponential with parameter k

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Kendall's notation

M	指數分佈 (馬可夫: **Markovian**)
D	退化 (**deterministic**) 分佈(或特定性分佈)
E$_k$	爾朗(**Erlang**)分佈 (*k* 為外型參數)
G	一般分佈(任意分佈)
H$_k$	參數為 *k* 之超指數

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Little's Law

- Assuming a queuing environment to be operating in a stable steady state where all initial transients have vanished, the key parameters characterizing the system are:
  - λ – the mean steady state consumer arrival
  - N – the average no. of customers in the system
  - T – the mean time spent by each customer in the system

which gives

$$N = \lambda T$$

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Markov Process

- A Markov process is one in which the next state of the process depends only on the present state, irrespective of any previous states taken by the process
- The knowledge of the current state and the transition probabilities from this state allows us to predict the next state

Russian mathematician Andrey Markov.

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Birth-Death Process

- Special type of Markov process
- Often used to model a population (or, no. of jobs in a queue)
- If, at some time, the population has *n* entities (*n* jobs in a queue), then birth of another entity (arrival of another job) causes the state to change to $n+1$
- On the other hand, a death (a job removed from the queue for service) would cause the state to change to $n-1$
- Any state transitions can be made only to one of the two neighboring states

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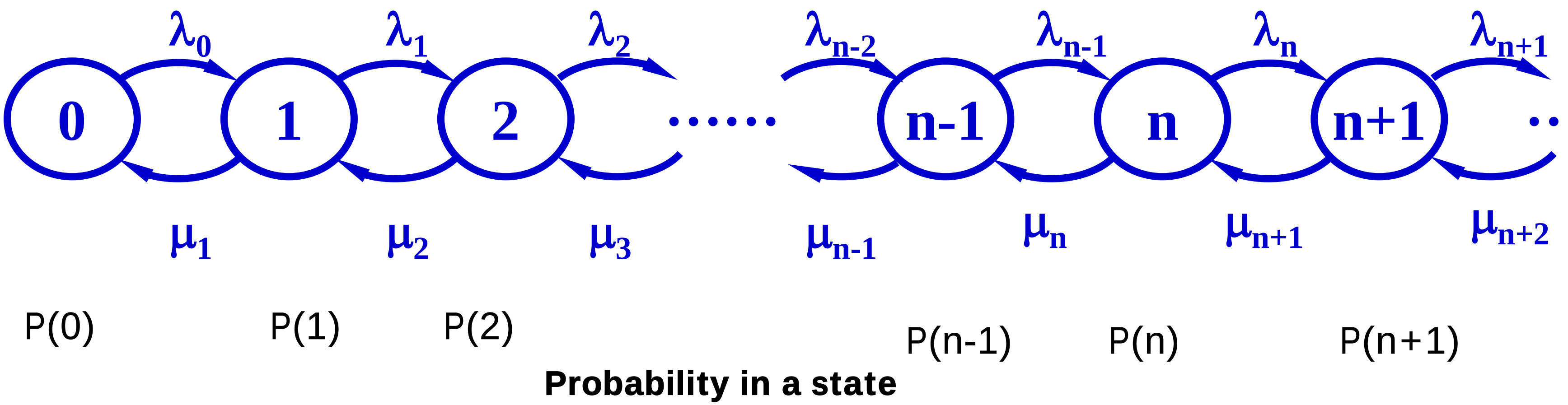
State Transition Diagram



The state transition diagram of the continuous birth-death process

- 狀態 0 的平衡方程式：$\mu_1 p_1 = \lambda_0 p_0$
- 狀態 1 的平衡方程式：$\lambda_0 p_0 + \mu_2 p_2 = (\lambda_1 + \mu_1) p_1$
- 狀態 n 的平衡方程式：$\lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1} = (\lambda_n + \mu_n) p_n$

Equilibrium State Equations

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Birth-Death Process

- 由平衡方程式可得

$$p_1 = \frac{\lambda_0}{\mu_1} p_0 \qquad p_2 = \frac{\lambda_1}{\mu_2} p_1 + \frac{1}{\mu_2}(\mu_1 p_1 - \lambda_0 p_0)$$

$$= \frac{\lambda_1}{\mu_2} p_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} p_0$$

- 由數學歸納法（mathematical induction）可得

$$p_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n \mu_{n-1}\cdots\mu_1} p_0$$

$$= p_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \qquad n = 1, 2, \cdots \qquad (3)$$

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Birth-Death Process

- 因為

$$\sum_{n=0}^{\infty} p_n = p_0 + p_0 \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} = 1$$

- 所以

$$p_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}} \qquad (4)$$

- 以下所討論的等候模式，若屬生死過程，即可用式(3)及式(4)

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M/M/1/∞ or M/M/1 Queuing System

- **When a customer arrives in this system it will be served if the server is free, otherwise the customer is queued**
- **In this system customers arrive according to a *Poisson distribution* and compete for the service in a FIFO (first in first out) manner**
- **Service times are independent identically distributed (IID) random variables, the common distribution being exponential**

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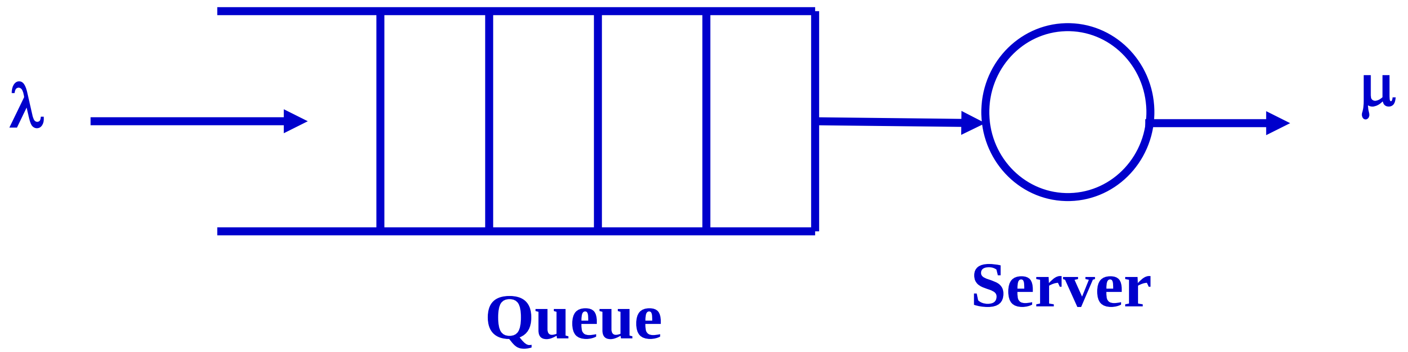
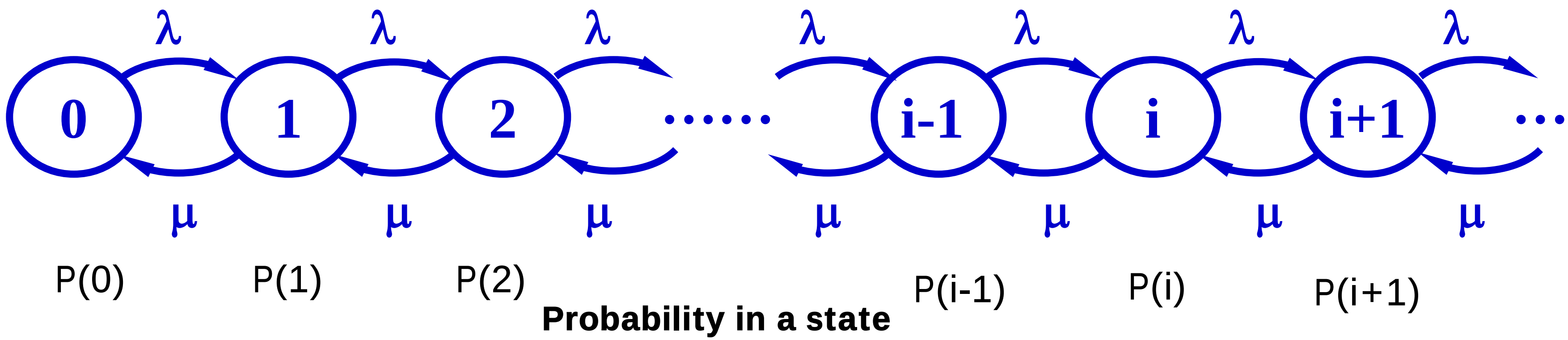
Queuing Model and State Transition Diagram



The M/M/1/∞ queuing model



The state transition diagram of the M/M/1/∞ queuing system

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Equilibrium State Equations

- **If mean arrival rate is λ and mean service rate is μ, i = 0, 1, 2 be the number of customers in the system and *P(i)* be the state probability of the system having *i* customers**
- **From the state transition diagram, the equilibrium state equations are given by**

$$\lambda P(0) = \mu P(1), \qquad i = 0,$$

$$(\lambda + \mu) P(i) = \lambda P(i-1) + \mu P(i+1), \qquad i \geq 1$$

$$P(i) = (\tfrac{\lambda}{\mu})^{i} P(0), \qquad i \geq 1$$

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Equilibrium State Equations

$$\begin{cases} P(1) = \dfrac{\lambda}{\mu} P(0), \\ P(2) = \dfrac{\lambda}{\mu} P(1) = (\dfrac{\lambda}{\mu})^2 P(0) = \rho^2 P(0), \\ \cdots \\ P(i) = \dfrac{\lambda}{\mu} P(i-1) = (\dfrac{\lambda}{\mu})^t P(0) = \rho^i P(0), \end{cases}$$

$$\rho = \frac{\lambda}{\mu}$$

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Traffic Intensity

- **We know that the *P(0)* is the probability of server being free. Since *P(0) > 0*, the necessary condition for a system being in steady state is,**

$$\rho = \frac{\lambda}{\mu} < 1$$

This means that the arrival rate cannot be more than the service rate, otherwise an infinite queue will form and jobs will experience infinite service time

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M/M/1 Queuing System

P_n為系統中顧客數 n 之機率，又機率總和等於 1，因此

$$\sum_{n=0}^{\infty} P_n = 1 \Rightarrow P_0 + P_1 + P_2 + \ldots + P_n + \ldots = 1$$

$$P_0 + (\lambda/\mu) P_0 + (\lambda/\mu)^2 P_0 + \ldots + (\lambda/\mu)^n P_0 + \ldots = 1$$

$$P_0 [\, 1 + (\lambda/\mu) + (\lambda/\mu)^2 + \ldots + (\lambda/\mu)^n + \ldots \,] = 1$$

Geometric Series (等比級數)

$$P_0 \left[\frac{1}{1-(\lambda/\mu)} \right] = 1$$

$$P_0 = 1 - (\lambda/\mu) = 1 - \rho, \quad \rho = \lambda/\mu$$

$$P_n = (\lambda/\mu)^n P_0 = \rho^n (1 - \rho) \quad , n \geq 0$$

P_0為系統中沒有顧客之機率，即服務設施閒置之機率。

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Queuing System Metrics

- $\rho = 1 - P(0)$, is the probability of the server being busy.
- Therefore, we have

$$P(n) = \rho^n (1 - \rho)$$

- The average number of customers in the system is

$$L_s = \frac{\lambda}{\mu - \lambda}$$

- The average dwell time of customers is

$$W_s = \frac{1}{\mu - \lambda}$$

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$$L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\,\rho^n (1 - \rho)$$

$$= \sum n\,\rho^n - \sum n\,\rho^{n+1}$$

$$= (0 + \rho + 2\rho^2 + 3\rho^3 + \ldots)$$

$$- (0 + \rho^2 + 2\rho^3 + \ldots)$$

$$= \rho + \rho^2 + \rho^3 + \ldots$$

$$= \rho\,(1 + \rho + \rho^2 + \rho^3 + \ldots)$$

$$= \rho \times \frac{1}{1 - \rho}$$

$$= \frac{\lambda}{\mu - \lambda}$$

(a)

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由(a)式 $$L = \lambda W$$

$$\frac{\lambda}{\mu - \lambda} = \lambda W$$

$$\therefore$$

$$W = \frac{1}{\mu - \lambda} \qquad \textbf{(c)}$$

由(c)式

$$W = W_q + 1/\mu$$

$$W_q = W - 1/\mu = \frac{\lambda}{\mu(\mu - \lambda)} \qquad \textbf{(b)}$$

由(b)式

$$L_q = \lambda W_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

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Queuing System Metrics

- The average queuing length is

$$Lq = \sum_{i=1}^{\infty} (i-1)P(i) = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

- The average waiting time of customers is

$$W_q = \frac{L_q}{\lambda} = \frac{\rho^2}{\lambda(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

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Example

- **問題**
  - 某郵局有一個專門辦理郵寄業務的窗口，中午12:00至下午1:00，到郵局辦理郵寄業務的顧客呈指數分配，平均每小時30人，每位顧客的服務時間亦呈指數分配，平均為1.5分鐘，求有五位顧客以上的機率為何?
- **解答**
  - 此窗口的等候模式為 $M/M/1$ 模式，且

$$\lambda = 30/\text{hr}$$

$$\frac{1}{\mu} = 1.5\,\text{min} = \frac{1}{40}\,\text{hr} \Rightarrow \mu = 40/\text{hr}$$

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Example

- 因此

$$\rho = \frac{\lambda}{\mu} = \frac{30}{40} = 0.75$$

- 代入$M/M/1$模式的相關公式可得

$$L = \frac{\lambda}{\mu - \lambda} = \frac{30}{40 - 30} = 3 \text{位顧客}$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{30^2}{40(40 - 30)} = 2.25 \text{位顧客}$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{40 - 30} = 0.1 \text{ hr} = 6 \text{ min}$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{30}{40(40 - 30)} = 0.075 \text{ hr} = 4.5 \text{ min}$$

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Example

- 代入 p_n 公式可得

$$p_n = \rho^n(1-\rho) = (0.75)^n(1-0.75)$$

- 因此

$$p_0 = 0.250,\ p_1 = 0.188,\ p_2 = 0.141,\ p_3 = 0.105,\ p_4 = 0.079$$

- 五位以上顧客的機率為

$$1-\sum_{n=0}^{4} p_n = 1-0.763 = 0.237$$

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M/M/S/∞

- Customers arrive according to a Poisson process with rate λ.
- The service times of customers are exponentially distributed with parameter μ.
- There are s servers, serving customers in order of arrival.
- Stability condition: $\lambda < s * \mu$ or alternatively written, $\rho = \lambda / (s * \mu) < 1$.

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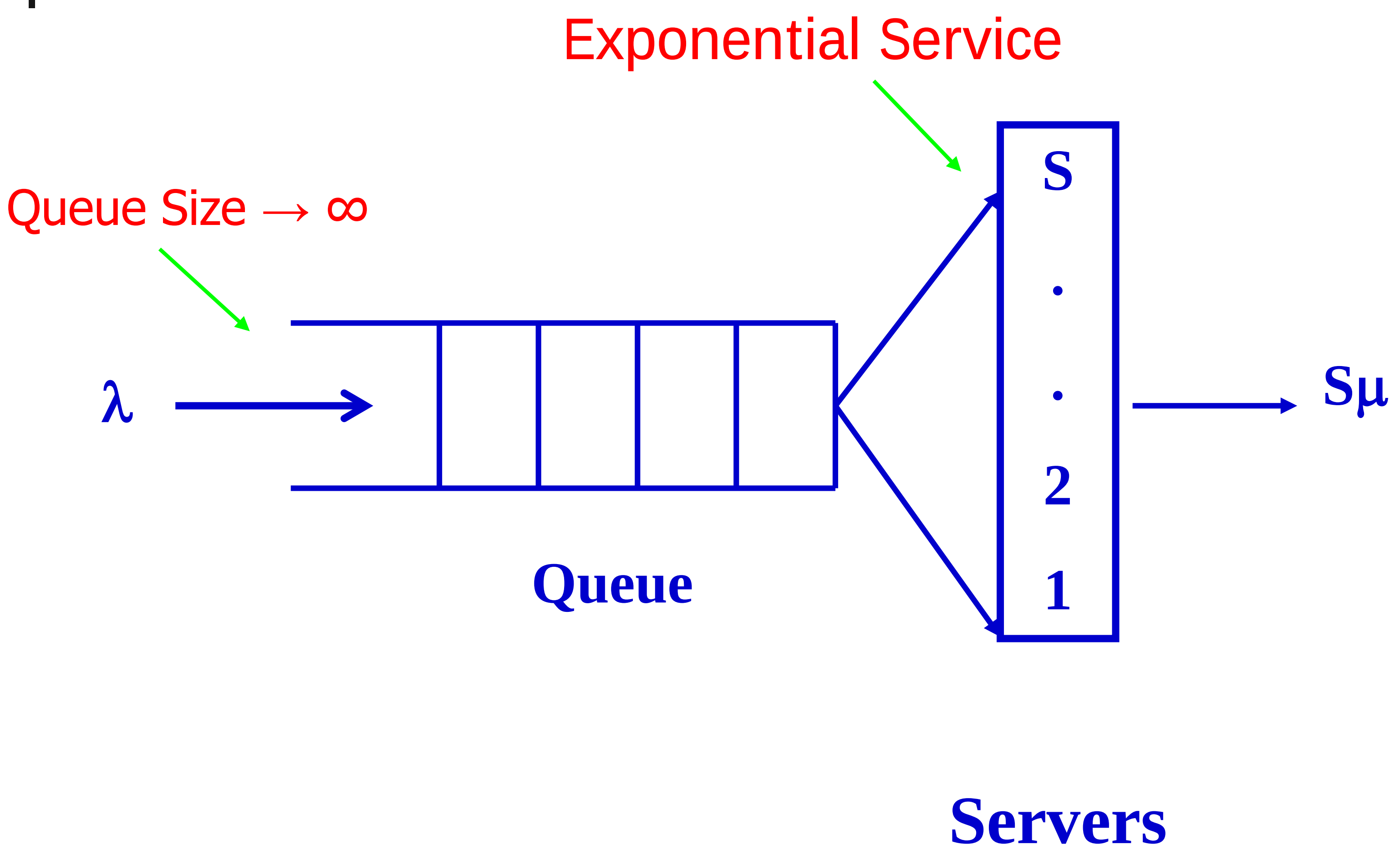
M/M/S/∞ Queuing Model



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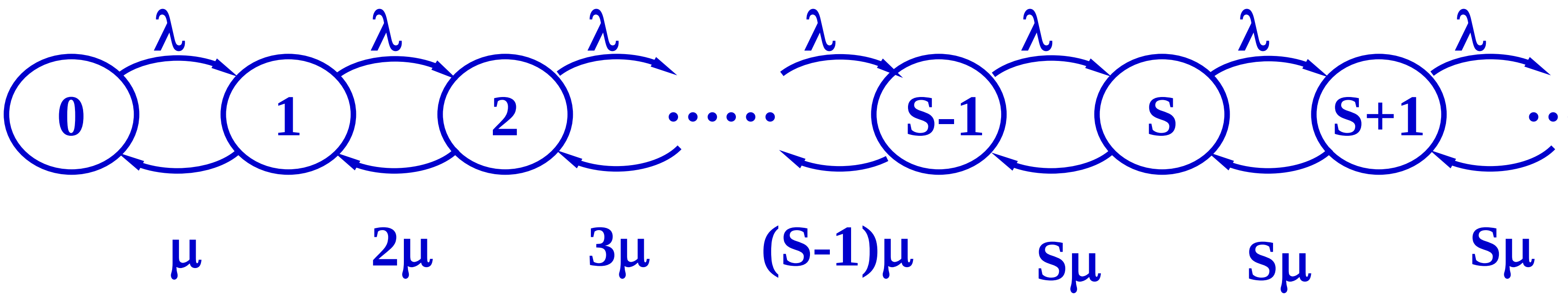
State Transition Diagram



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M/M/S/∞ Queuing Model

$$p_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} p_0$$

$$= p_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \qquad n = 1, 2, \cdots$$

- 此模式的 λ_n 及 μ_n 如下：

$$\lambda_n = \lambda \qquad n = 0, 1, \ldots$$

$$\mu_n = \begin{cases} n\mu & n = 1, 2, \ldots, s-1 \\ s\mu & n = s, s+1, \ldots \end{cases}$$

- 欲求得 p_n，我們先計算式(3)中的乘積部分如下：

$$\prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} = \begin{cases} \dfrac{\lambda^n}{(1\mu)(2\mu)\cdots(n\mu)} = \dfrac{(\lambda/\mu)^n}{n!} & n = 1, 2, \ldots, s-1 \\[2ex] \dfrac{\lambda^n}{(1\mu)(2\mu)\cdots(s\mu)(s\mu)^{n-s}} = \dfrac{(\lambda/\mu)^n}{s!s^{n-s}} & n = s, s+1, \ldots \end{cases}$$

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M/M/S/∞ Queuing Model

- Thus，

$$p_n = \begin{cases} \dfrac{(\lambda/\mu)^n}{n!} p_0 & n = 1, 2, \ldots, s-1 \\ \dfrac{(\lambda/\mu)^n}{s!s^{n-s}} p_0 & n = s, s+1, \ldots \end{cases}$$

$$p_0 = \left[1 + \sum_{n=1}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!} \sum_{n=s}^{\infty} \left(\frac{\lambda}{s\mu} \right)^{n-s} \right]^{-1}$$

$$= \left[\sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!} \frac{1}{1-\lambda/(s\mu)} \right]^{-1}$$

Geometric Series (等比級數)

- 以上 p_0 的求導過程中，必須 $\rho = \lambda / s\mu < 1$

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M/M/S/∞

- 至於四個績效基準，可計算 L_q 如下：

$$L_q = \sum_{n=s}^{\infty}(n-s)p_n$$

$$= \sum_{i=0}^{\infty} i p_{s+i} = \sum_{i=0}^{\infty} i \frac{(\lambda/\mu)^s}{s!}\rho^i p_0$$

$$= p_0 \frac{(\lambda/\mu)^s}{s!}\rho \sum_{i=0}^{\infty} i\rho^{i-1} = p_0 \frac{(\lambda/\mu)^s}{s!}\rho \sum_{i=0}^{\infty} \frac{d}{d\rho}\rho^i$$

$$= p_0 \frac{(\lambda/\mu)^s}{s!}\rho \frac{d}{d\rho}\sum_{i=0}^{\infty}\rho^i = p_0 \frac{(\lambda/\mu)^s}{s!}\rho \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

$$= \frac{(\lambda/\mu)^s \rho}{s!(1-\rho)^2} p_0$$

- 其餘的績效基準可由 Little 公式及基本關係求得

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Queuing System Metrics

- **The average number of customers in the system is**

$$L = \lambda(W_q + \frac{1}{\mu}) = L_q + \frac{\lambda}{\mu}$$

$$\frac{(\lambda/\mu)^s \rho}{s!(1-\rho)^2} p_0$$

- **The average waiting time of customers is**

$$W_q = \frac{L_q}{\lambda}$$

- **The average dwell time of a customer in the system is given by**

$$W = W_q + \frac{1}{\mu}$$

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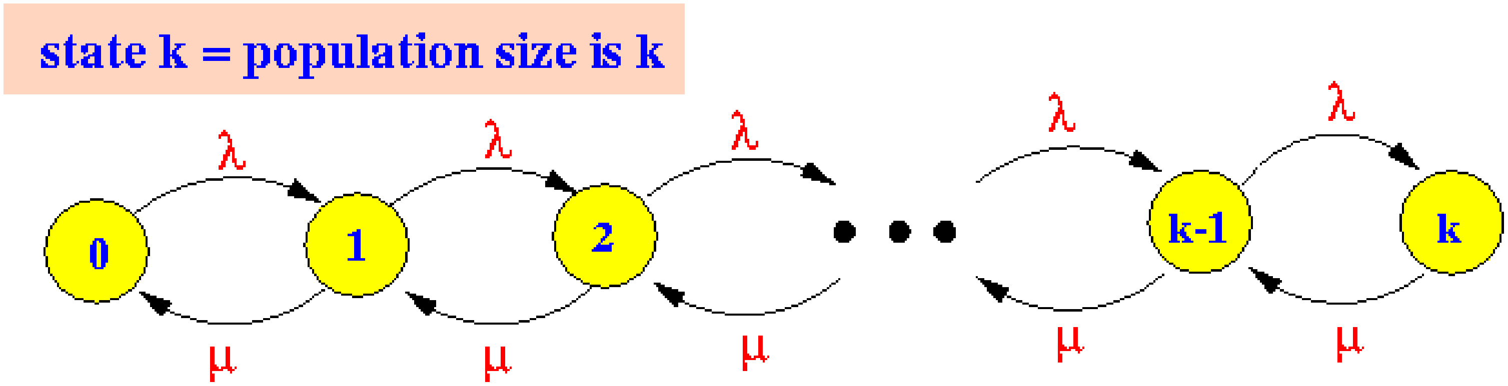
M/M/1/K

- Arrival process is Poisson Process with rate λ
- Number of services is Poisson Process with rate μ
- Single Server
- Queue size is finite = K
- Queue discipline : FCFS



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M/M/1/K

- 到達率與服務率為

$$\lambda_n = \begin{cases} \lambda & n = 0,1,\dots,K-1 \\ 0 & n = K, K+1,\dots \end{cases}$$

$$\mu_n = \mu \qquad n = 1,2,\cdots$$

$$p_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} p_0$$

$$= p_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \qquad n = 1,2,\cdots \qquad (3)$$

$$p_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}} \qquad (4)$$

- 將 λ_n 與 μ_n 代入式(4)及式(3)，分別可得

$$p_0 = \frac{1}{\rho^0 + \sum_{n=1}^{K} \rho^n} = \frac{1}{\sum_{n=0}^{K} \rho^n}$$

$$= \begin{cases} \dfrac{1-\rho}{1-\rho^{K+1}} & \rho \neq 1 \\ \dfrac{1}{K+1} & \rho = 1 \end{cases}$$

$$P_n = \rho^n \frac{1-\rho}{1-\rho^{k+1}}$$

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M/M/1/K

$$p_n = \rho^n p_0$$

$$= \begin{cases} \dfrac{1-\rho}{1-\rho^{K+1}}\rho^n & \rho \neq 1 \\ \dfrac{1}{K+1} & \rho = 1 \end{cases} \quad \text{for } n = 0,1,\ldots,K$$

- 在以上 p_0 推導中，用到有限總和（finite sum）公式：

$$\sum_{n=0}^{K} x^n = \begin{cases} \dfrac{1-x^{K+1}}{1-x} & x \neq 1 \\ K+1 & x = 1 \end{cases}$$

For $r \neq 1$, the sum of the first n terms of a geometric series is

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\,\frac{1-}{1-}$$

- 此模式的穩定狀態並不需要 $\rho = \lambda / \mu < 1$

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M/M/1/K

$$
\begin{aligned}
L &= \sum_{k=1}^{K} k\rho^k P_0 = \rho P_0 \sum_{k=1}^{K} k\rho^{k-1} \\
&= \rho P_0 \left(\sum_{k=1}^{K} \rho^k\right)' = \rho P_0 \left(\rho\frac{1-\rho^K}{1-\rho}\right)' = \rho P_0 \left(\frac{\rho-\rho^{K+1}}{1-\rho}\right)' \\
&= \left(\left(1-(K+1)\rho^K\right)(1-\rho)+\rho-\rho^{K+1}\right)\cdot\frac{\rho P_0}{(1-\rho)^2} \\
&= \frac{\rho P_0\left(1-(K+1)\rho^K-\rho+(K+1)\rho^{K+1}+\rho-\rho^{K+1}\right)}{(1-\rho)^2} \\
&= \frac{\rho P_0\left(1-(K+1)\rho^K+K\rho^{K+1}\right)}{(1-\rho)^2} \\
&= \frac{\rho\left(1-(K+1)\rho^K+K\rho^{K+1}\right)}{(1-\rho)(1-\rho^{K+1})}.
\end{aligned}
$$

Quotient rule

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(Quotient Rule) $\frac{d}{dx}\left(\frac{f}{g}\right)=\frac{\frac{df}{dx}\cdot g-f\cdot\frac{dg}{dx}}{g^2}$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{\frac{df(x)}{dx}\cdot g(x)-f(x)\cdot\frac{dg(x)}{dx}}{g^2(x)}$$

$$=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g^2(x)}$$

$$\frac{1}{(1-\rho)^2}\times\left((1-\rho)\times\left(1-(k+1)\rho^k\right)-\left(\rho-\rho^{k+1}\right)\times(-1)\right)$$

$$\frac{1}{(1-\rho)^2}\times\left((1-\rho)\times\frac{d}{d\rho}\left(\rho-\rho^{k+1}\right)-\left(\rho-\rho^{(k+1)}\right)\times\frac{d}{d\rho}(1-\rho)\right)$$

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M/M/1/K

- 當 $\rho \neq 1$ 時：

$$L = \ldots$$

$$= \frac{\rho[K\rho^{K+1} - (K+1)\rho^{K} + 1]}{(1-\rho^{K+1})(1-\rho)}$$

- 若 $\rho = 1$，則

$$L = \sum_{n=0}^{K} n p_n = \frac{1}{K+1} \frac{K(K+1)}{2} = \frac{K}{2}$$

- L_q 可計算如下：

$$L_q = L - (1 - p_0)$$

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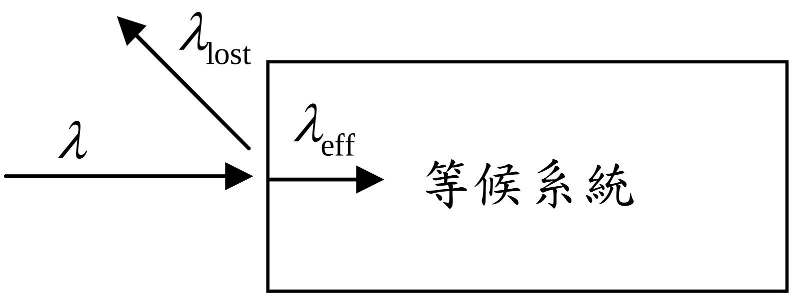
M/M/1/K

- 利用 Little 公式時，須使用有效到達率(effective arrival rate）λ_{eff} ：

$$\lambda_{\text{eff}} = \lambda(1 - p_K)$$

- 代入 Little 公式可得

$$W = \frac{L}{\lambda_{\text{eff}}} \qquad W_q = \frac{L_q}{\lambda_{\text{eff}}}$$



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M/M/S/K

- 到達率與服務率為：

$$\lambda_n = \begin{cases} \lambda & n = 0, 1, \ldots, K-1 \\ 0 & n = K, K+1, \ldots \end{cases}$$

$$\mu_n = \begin{cases} n\mu & n = 1, 2, \ldots, s-1 \\ s\mu & n = s, s+1, \ldots, K \end{cases}$$

- 將 λ_n 與 μ_n 代入式(3)及式(4)，分別可得

$$p_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} p_0 = p_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \qquad n = 1, 2, \cdots \qquad (3)$$

$$p_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}} \qquad (4)$$

$$p_n = \begin{cases} \frac{(\lambda/\mu)^n}{n!} p_0 & n = 1, 2, \ldots, s-1 \\ \frac{(\lambda/\mu)^n}{s!s^{n-s}} p_0 & n = s, s+1, \ldots, K \\ 0 & n = K+1, \ldots \end{cases}$$

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M/M/S/K

For $r \neq 1$, the sum of the first n terms of a geometric series is

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r},$$

$$p_0 = \left[1 + \sum_{n=1}^{s-1}\frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!}\sum_{n=s}^{K}\left(\frac{\lambda}{s\mu}\right)^{n-s}\right]^{-1}$$

$$= \left[1 + \sum_{n=1}^{s-1}\frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!}\frac{1-(\lambda/s\mu)^{K-s+1}}{1-(\lambda/s\mu)}\right]^{-1}$$

- 得到所有 p_n 後，即可求得 L_q（當 $\rho = \lambda/s\mu \neq 1$ 時）如下：

$$L_q = \sum_{n=s}^{K}(n-s)p_n = \ldots$$

$$= \frac{(\lambda/\mu)^s \rho}{s!(1-\rho)^2} p_0[(K-s)(\rho-1)\rho^{K-s} - \rho^{K-s} + 1]$$

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M/M/S/K

- 此模式之 L 與 L_q 的關係如下：

$$L = L_q + \sum_{n=0}^{s-1} np_n + s\left(1 - \sum_{n=0}^{s-1} p_n\right)$$

- 欲利用 Little 公式，須使用有效到達率 λ_{eff} ：

$$\lambda_{\text{eff}} = \lambda(1 - p_K)$$

- 因此，

$$W = \frac{L}{\lambda_{\text{eff}}} \qquad W_q = \frac{L_q}{\lambda_{\text{eff}}}$$

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Example

- **概述**
- 保養廠設置2個升降工作台（各1位維修員），並可停放3輛
- 若汽車無法進入保養廠停放，將會離開
- 到達率呈指數分配，平均每小時2輛
- 維修時間呈指數分配，平均需要40分鐘
- 每位顧客平均消費金額$1350

- **問題**

a. 廠內有n位顧客的機率
b. 有效到達率
c. 每天營業的10小時期間，因顧客無法進入而損失的營業額
d. 兩工作台的期望車輛數
e. 每位維修員每天空閒時間的百分比
f. 等候維修的期望車輛數
g. 每位顧客在保養廠內的期望時間

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Solution

(a)

- 此為 $M/M/s/K$ 模式，其 $s=2, K=5$，且

$$\lambda_n = 2/\text{hr} \qquad n = 0,1,\ldots,4$$

$$\frac{1}{\mu} = 40\,\text{min} = \frac{2}{3}\text{hr} \Rightarrow \mu = 1.5/\text{hr}$$

$$\mu_n = \begin{cases} 1.5n & n = 1,2 \\ 3 & n = 3,4,5 \end{cases}$$

- 代入 p_n 公式可得

$$p_n = \begin{cases} \dfrac{(2/1.5)^n}{n!} p_0 & n = 1,2 \\ \dfrac{(2/1.5)^n}{2!2^{n-s}} p_0 & n = 3,4,5 \end{cases}$$

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Solution

(a)

- 因此，

$$p_1 = \frac{4}{3} p_0,\ p_2 = \frac{1}{2}\left(\frac{4}{3}\right)^2 p_0,\ p_3 = \frac{1}{4}\left(\frac{4}{3}\right)^3 p_0$$

$$p_4 = \frac{1}{8}\left(\frac{4}{3}\right)^4 p_0,\ p_5 = \frac{1}{16}\left(\frac{4}{3}\right)^5 p_0$$

- 因機率總和等於1，所以

$$p_0\left(1+\frac{4}{3}+\frac{1}{2}\left(\frac{4}{3}\right)^2+\frac{1}{4}\left(\frac{4}{3}\right)^3+\frac{1}{8}\left(\frac{4}{3}\right)^4+\frac{1}{16}\left(\frac{4}{3}\right)^5\right)=1$$

- 計算可得 $p_0 = 0.2236$，因此

$$p_1 = 0.2981,\ p_2 = 0.1988,\ p_3 = 0.1325,\ p_4 = 0.0883,\ p_5 = 0.058$$

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Solution

(b)

- 有效到達率可計算如下：

$$\lambda_{\text{eff}} = \lambda(1 - p_5) = 2(1 - 0.0589) = 1.8822 \text{ cars / hr}$$

(c)

- 每小時無法進入的車輛數為

$$\lambda p_5 = 2(0.0589) = 0.1178 \quad \text{cars}$$

- 因此每天 10 小時所損失的營業額為

$$\$1350 \times 10 \times \lambda p_5 = \$1590.3$$

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Solution

(d) 兩工作台的期望車輛數為

$$1p_1 + 2(p_2 + p_3 + p_4 + p_5) = 1.2551 \text{ cars}$$

(e) 每位維修員每天空閒時間的百分比可計算如下：

$$(2p_0 + 1p_1)/2 = 37.27\%$$

(f) 等候維修的期望車輛數為

$$L_q = \sum_{n=3}^{5}(n-2)p_n = 1p_3 + 2p_4 + 3p_5 = 0.4858 \text{ cars}$$

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Solution

(g) 每位顧客在保養廠內的期望時間為

$$W_q = \frac{L_q}{\lambda_{\text{eff}}} = \frac{0.4858}{1.8822} = 0.2581 \text{ hrs} = 15.49 \text{ min}$$

$$W = W_q + \frac{1}{\mu} = 55.49 \text{ min}$$

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Example

- 某工廠買了許多同一型式之機器，現在需要確定一名工人應看管幾台機器，機器在正常運轉時是不需要看管的。已知每台機器的正常運轉時間服從平均數為120分鐘的指數分配，工人看管一台機器的時間服從平均數為12分鐘的指數分配。每名工人只能看管自己的機器，工廠要求每台機器的正常運轉時間不得少於87.5％。問在此條件之下每名工人最多能看管幾台機器？

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Solution

Sol:這是M/M/1,有限來源排隊系統。每名工人看管最多台數爲 k

$\lambda = \frac{1}{2}, \mu = 5$

$\therefore \rho = \frac{1}{10}$

因此，

$$L_s = k - \frac{\mu}{\lambda}(1 - P_0) = k - \rho(1 - P_0)$$

$$P_0 = \left[\sum_{n=0}^{k} \frac{k!}{(k-n)!} \rho^n \right]^{-1}$$

根據要求，停止運轉的機器數 $L_s \le 0.125k$。當m=1,2,3,4,5,時如以下之結果：

k	P_0	L_s	$L_s \le 0.125k$?
1	10/11	0.091	是(0.125)
2	50/61	0.197	是(0.25)
3	500/683	0.321	是(0.375)
4	1250/1933	0.467	是(0.5)
5	2500/4433	0.640	否(0.625)

可見一名工人最多只能看管四台機器

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Example

- 某戲院有三個售票窗口，每個售票窗口對於每位客人之平均出售時間為5分鐘，依指數分佈；到達該戲院每小時平均6人，依Poisson分佈。但等候列的長度為30個人，試求下列問題？

（１） 系列中有n個客人的機率

（２） 系列之呼損率（系列中已滿，再到達之客人不能進入系列中而馬上離開之機率）

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Solution

Sol:此題爲M/M/3/30，S=3，K=30，$\lambda=6$（人/小時）

$$\mu=\tfrac{60}{5}(人/分)=12(人/小時)$$

$$\rho=\tfrac{\lambda}{3\mu}=\tfrac{1}{6}<1, 先求P_0$$

$$\sum_{n=0}^{3}\frac{1}{n!}(1/2)^n+\frac{3^3}{3!}\sum_{n=4}^{30}(1/6)^n$$

$$=1+\left(\tfrac{1}{2}\right)+\tfrac{1}{2}\left(\tfrac{1}{2}\right)^2+\tfrac{1}{6}\left(\tfrac{1}{2}\right)^3+\tfrac{9}{2}\times\frac{\left(\tfrac{1}{6}\right)^4\left[1-\left(\tfrac{1}{6}\right)^{27}\right]}{1-\tfrac{1}{6}}$$

$$=1.65$$

$$P_0=\tfrac{1}{1.65}=0.61$$

$$（1）\ P_n=\begin{cases}\dfrac{1}{n!}\left(\dfrac{1}{6}\right)^n\times 0.61, & 0\le n<3\\ \dfrac{1}{3!3^{n-3}}\left(\dfrac{1}{6}\right)^n\times 0.61, & 3\le n<30\end{cases}$$

$$（2）\ P_N=\frac{\rho^N}{s!s^{N-s}}P_0=\frac{1}{3!3^{27}}\left(\frac{1}{6}\right)^{30}\times 0.61=0$$

HW

- 有一超級市場在顧客的期望下，該店經理相信顧客對需排隊等候8分鐘並且共花10分鐘在等候系統上（不含真正購物時間）是難以接受的。因此經理嘗試下列兩方案中的一個來縮短顧客等待時間：（1）增聘一位員工打包貨品；（2）增加一個結帳櫃臺。

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