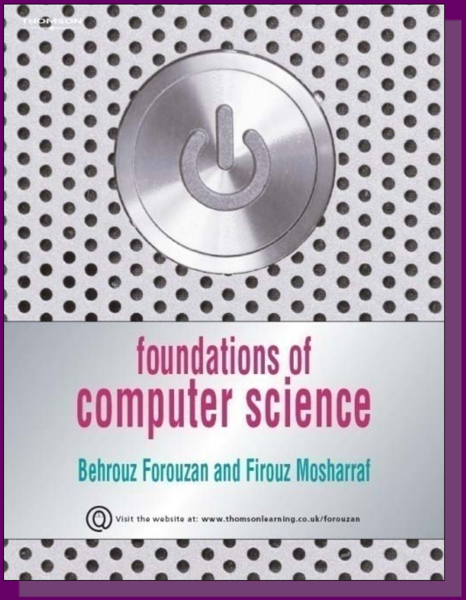


2
Number Systems

Objectives

After studying this chapter, the student should be able to:

- ❑ Understand the concept of number systems.
- ❑ Distinguish between non-positional and positional number systems.
- ❑ Describe the decimal, binary, hexadecimal and octal system.
- ❑ Convert a number in binary, octal or hexadecimal to a number in the decimal system.
- ❑ Convert a number in the decimal system to a number in binary, octal and hexadecimal.
- ❑ Convert a number in binary to octal and vice versa.
- ❑ Convert a number in binary to hexadecimal and vice versa.
- ❑ Find the number of digits needed in each system to represent a particular value.

2-1 INTRODUCTION

A **number system** defines how a number can be represented using distinct symbols.

A number can be represented differently in different systems. For example, the two numbers $(2A)_{16}$ and $(52)_8$ both refer to the same quantity, $(42)_{10}$, but their representations are different.

Several number systems have been used in the past and can be categorized into two groups: **positional** and **non-positional** systems.

Our main goal is to discuss the positional number systems, but we also give examples of non-positional systems.

2-2 POSITIONAL NUMBER SYSTEMS

In a **positional number system**, the position a symbol occupies in the number determines the value it represents. In this system, a number represented as:

$$\boxed{\pm (S_{k-1} \ldots S_2\, S_1\, S_0 .\, S_{-1}\, S_{-2} \ldots S_{-l})_b}$$

has the value of:

$$n = \pm \; S_{k-1} \times b^{k-1} + \ldots + S_1 \times b^1 + S_0 \times b^0 \; + \; S_{-1} \times b^{-1} + S_{-2} \times b^{-2} + \ldots + S_{-l} \times b^{-l}$$

in which **S is the set of symbols**, **b is the base** (or **radix**).

The decimal system (base 10)

The word decimal is derived from the Latin root **decem** (ten). In this system the **base b = 10** and we use ten symbols

$$\boxed{S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}}$$

The symbols in this system are often referred to as **decimal digits** or just **digits**.

Integers

$$N = \pm \; S_{k-1} \times 10^{k-1} + S_{k-2} \times 10^{k-2} + \ldots + S_2 \times 10^2 + S_1 \times 10^1 + S_0 \times 10^0$$

	10^{k-1}	10^{k-2}	$\cdots$	10^2	10^1	10^0	Place values
$\pm$	S_{k-1}	S_{k-2}	$\cdots$	S_2	S_1	S_0	Number
	$\downarrow$	$\downarrow$		$\downarrow$	$\downarrow$	$\downarrow$	

$$N = \pm\; S_{k-1} \times 10^{k-1} + S_{k-2} \times 10^{k-2} + \cdots + S_2 \times 10^2 + S_1 \times 10^1 + S_0 \times 10^0$$ Values

Figure 2.1 **Place values for an integer in the decimal system**

Example 2.1

The following shows the place values for the integer +224 in the decimal system.

			10^2		10^1		10^0	Place values
			2		2		4	Number
N	=	+	2×10^2	+	2×10^1	+	4×10^0	Values

Note that the digit 2 in position 1 has the value 20, but the same digit in position 2 has the value 200. Also note that we normally drop the plus sign, but it is implicit.

Example 2.2

The following shows the place values for the decimal number −7508. We have used 1, 10, 100, and 1000 instead of powers of 10.

1000	100	10	1	Place values
7	5	0	8	Number

$$N = -(7 \times 1000 + 5 \times 100 + 0 \times 10 + 8 \times 1) \quad \textbf{Values}$$

Reals

Integral part		Fractional part
$R = \pm \; S_{k-1} \times 10^{k-1} + \ldots + S_1 \times 10^1 + S_0 \times 10^0$	$+$	$S_{-1} \times 10^{-1} + \ldots + S_{-l} \times 10^{-l}$

Example 2.3

The following shows the place values for the real number +24.13.

	10^1		10^0		10^{-1}	10^{-2}	Place values
	2		4	•	1	3	Number
$R = +$	2×10	$+$	4×1	$+$	1×0.1 $+$	3×0.01	Values

The binary system (base 2)

The word binary is derived from the Latin root **bini** (or two by two). In this system the **base b = 2** and we use only two symbols,

$$\boxed{\mathbf{S = \{0, 1\}}}$$

The symbols in this system are often referred to as **binary digits** or **bits** (binary digit).

Integers

$$N = \pm\ S_{k-1} \times 2^{k-1} + S_{k-2} \times 2^{k-2} + \ldots + S_2 \times 2^2 + S_1 \times 2^1 + S_0 \times 2^0$$

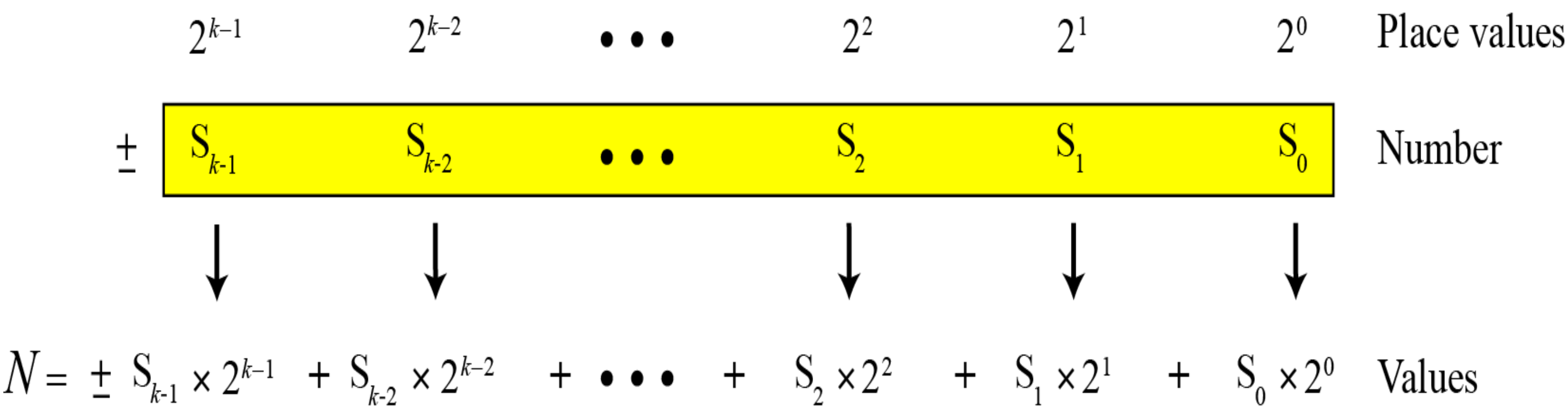


Figure 2.2 **Place values for an integer in the binary system**

Example 2.4

The following shows that the number $(11001)_2$ in binary is the same as 25 in decimal. The subscript 2 shows that the base is 2.

		2^4		2^3		2^2		2^1		2^0	Place values
		1		1		0		0		1	Number
N	=	1×2^4	+	1×2^3	+	0×2^2	+	0×2^1	+	1×2^0	Decimal

The equivalent decimal number is N = 16 + 8 + 0 + 0 + 1 = 25.

Reals

$$\text{Integral part} \quad \bullet \quad \text{Fractional part}$$

$$R = \pm \quad S_{k-1} \times 2^{k-1} + \ldots + S_1 \times 2^1 + S_0 \times 2^0 \quad + \quad S_{-1} \times 2^{-1} + \ldots + S_{-l} \times 2^{-l}$$

Example 2.5

The following shows that the number $(101.11)_2$ in binary is equal to the number 5.75 in decimal.

		2^2		2^1		2^0		2^{-1}		2^{-2}	Place values
		1		0		1	$\bullet$	1		1	Number
R	=	1×2^2	+	0×2^1	+	1×2^0	+	1×2^{-1}	+	1×2^{-2}	Values

The hexadecimal system (base 16)

The word **hexadecimal** is derived from the Greek root **hex** (six) and the Latin root **decem** (ten). In this system the **base b = 16** and we use sixteen symbols to represent a number. The set of symbols is

$$\boxed{S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F\}}$$

Note that the symbols A, B, C, D, E, F are equivalent to 10, 11, 12, 13, 14, and 15 respectively.

The symbols in this system are often referred to as **hexadecimal digits**.

Integers

$$N = \pm S_{k-1} \times 16^{k-1} + S_{k-2} \times 16^{k-2} + \ldots + S_2 \times 16^2 + S_1 \times 16^1 + S_0 \times 16^0$$

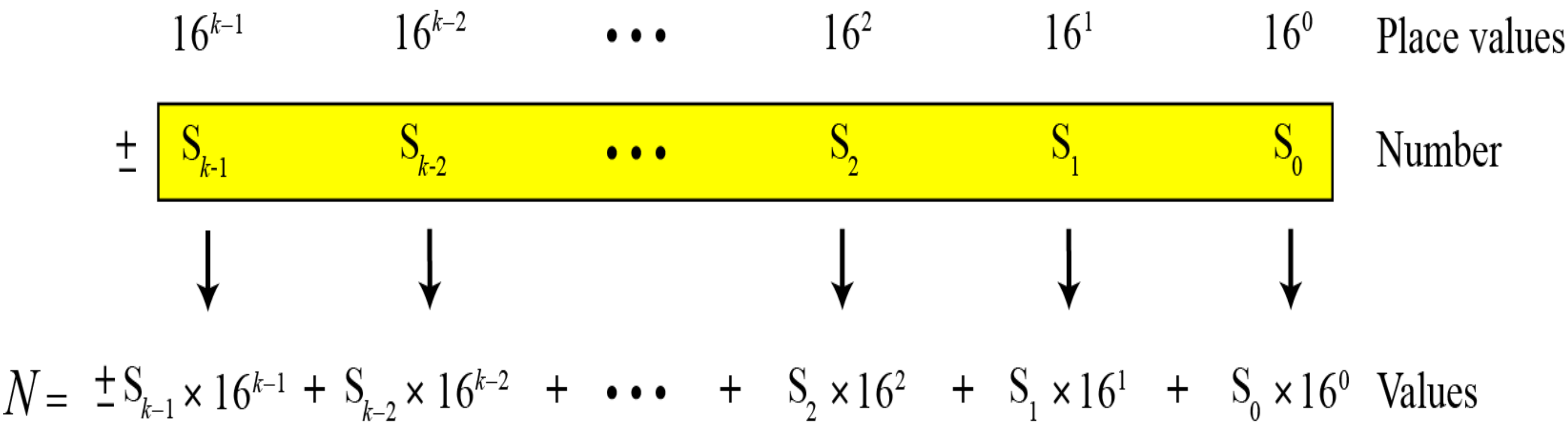


Figure 2.3 **Place values for an integer in the hexadecimal system**

Example 2.6

The following shows that the number $(2AE)_{16}$ in hexadecimal is equivalent to 686 in decimal.

		16^2		16^1		16^0	Place values
		2		A		E	Number
N	=	2×16^2	+	10×16^1	+	14×16^0	Values

The equivalent decimal number is N = 512 + 160 + 14 = 686.

The octal system (base 8)

The word octal is derived from the Latin root **octo** (eight). In this system the **base b = 8** and we use eight symbols to represent a number. The set of symbols is

$$\boxed{\mathbf{S = \{0, 1, 2, 3, 4, 5, 6, 7\}}}$$

Integers

$$N = \pm S_{k-1} \times 8^{k-1} + S_{k-2} \times 8^{k-2} + \ldots + S_2 \times 8^2 + S_1 \times 8^1 + S_0 \times 8^0$$

	8^{k-1}	8^{k-2}	$\cdots$	8^2	8^1	8^0	Place values
$\pm$	S_{k-1}	S_{k-2}	$\cdots$	S_2	S_1	S_0	Number

$$N = \pm S_{k-1} \times 8^{k-1} + S_{k-2} \times 8^{k-2} + \cdots + S_2 \times 8^2 + S_1 \times 8^1 + S_0 \times 8^0 \quad \text{Values}$$

Figure 2.3 **Place values for an integer in the octal system**

Example 2.7

The following shows that the number $(1256)_8$ in octal is the same as 686 in decimal.

		8^3		8^2		8^1		8^0	Place values
		1		2		5		6	Number
N	=	1×8^3	+	2×8^2	+	5×8^1	+	6×8^0	Values

Note that the decimal number is N = 512 + 128 + 40 + 6 = 686.

Summary

Table 2.1 shows a summary of the **four** positional number systems discussed in this chapter.

Table 2.1 Summary of the four positional number systems

System	*Base*	*Symbols*	*Examples*
Decimal	10	0, 1, 2, 3, 4, 5, 6, 7, 8, 9	2345.56
Binary	2	0, 1	$(1001.11)_2$
Octal	8	0, 1, 2, 3, 4, 5, 6, 7	$(156.23)_8$
Hexadecimal	16	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F	$(A2C.A1)_{16}$

Table 2.2 shows how the number 0 to 15 is represented in different systems.

Table 2.2 Comparison of numbers in the four systems

Decimal	*Binary*	*Octal*	*Hexadecimal*
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	5	5
6	110	6	6
7	111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Conversion

We need to know how to convert a number in one system to the equivalent number in another system.

Since the decimal system is more familiar than the other systems, we first show how to covert from any base to decimal.

Then we show how to convert from decimal to any base. Finally, we show how we can easily convert from binary to hexadecimal or octal and vice versa.

Any base to decimal conversion

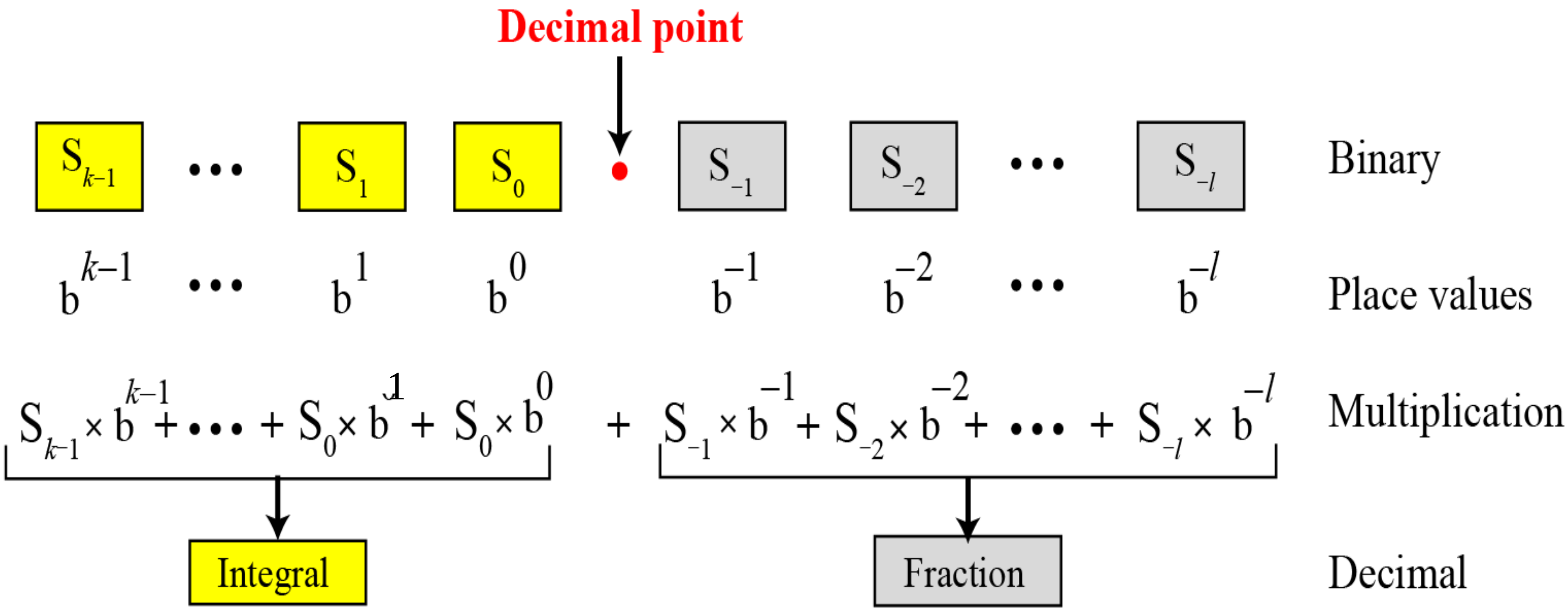


Figure 2.5 **Converting other bases to decimal**

Example 2.8

The following shows how to convert the binary number $(110.11)_2$ to decimal: $(110.11)_2 = 6.75$.

Binary	1		1		0	•	1		1
Place values	2^2		2^1		2^0		2^{-1}		2^{-2}
Partial results	4	+	2	+	0	+	0.5	+	0.25

Decimal: 6.75

Example 2.9

The following shows how to convert the hexadecimal number $(1A.23)_{16}$ to decimal.

Hexadecimal	1		A	•	2		3
Place values	16^1		16^0		16^{-1}		16^{-2}
Partial result	16	+	10	+	0.125	+	0.012

Decimal: 26.137

Note that the result in the decimal notation is not exact, because $3 \times 16^{-2} = 0.01171875$. We have rounded this value to three digits (0.012).

Example 2.10

The following shows how to convert $(23.17)_8$ to decimal.

Octal	2		3	•	1		7
Place values	8^1		8^0		8^{-1}		8^{-2}
Partial result	16	+	3	+	0.125	+	0.109

Decimal: 19.234

This means that $(23.17)_8 \approx 19.234$ in decimal. Again, we have rounded up $7 \times 8^{-2} = 0.109375$.

Decimal to any base

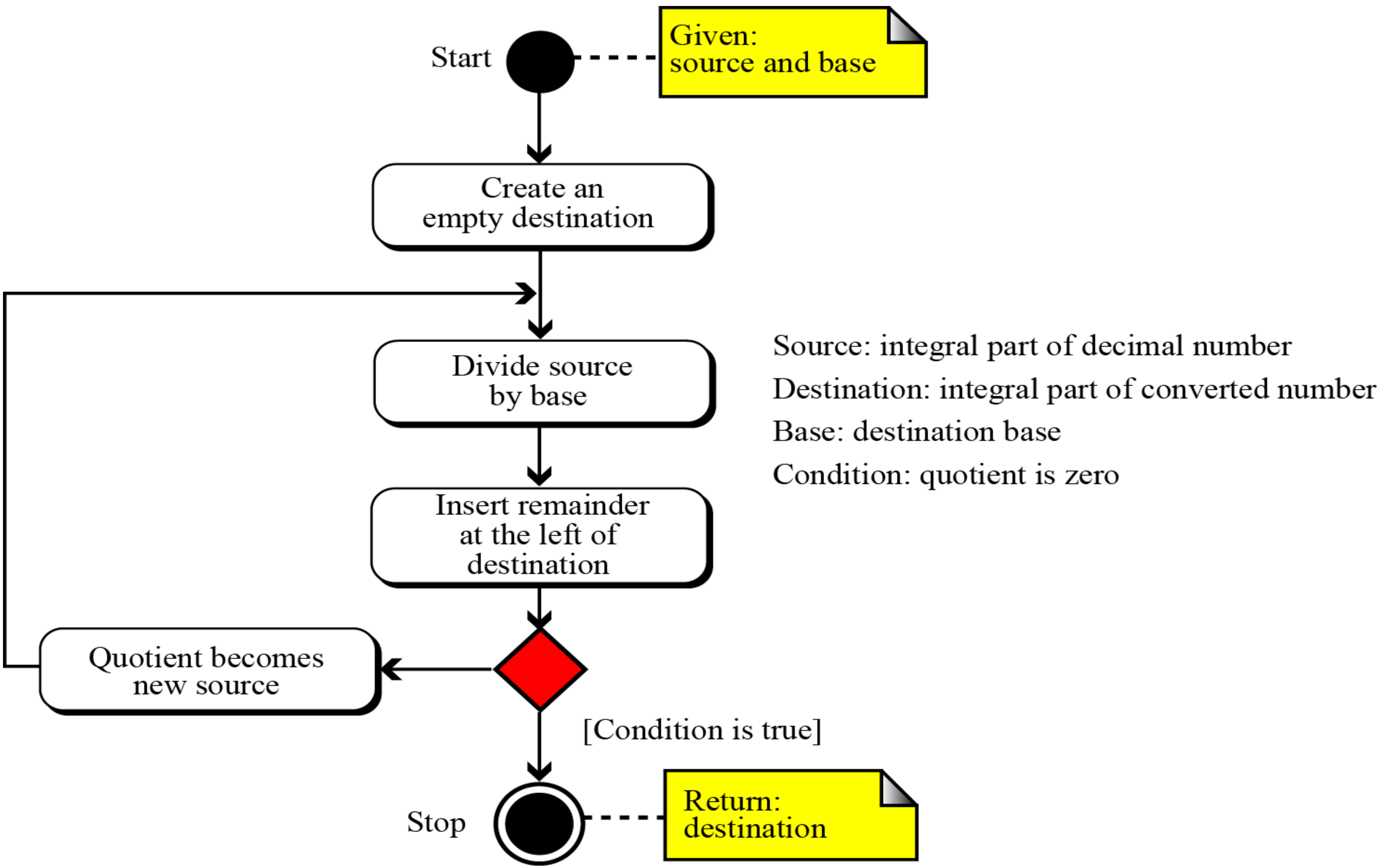


Figure 2.6 **Converting other bases to decimal (integral part)**

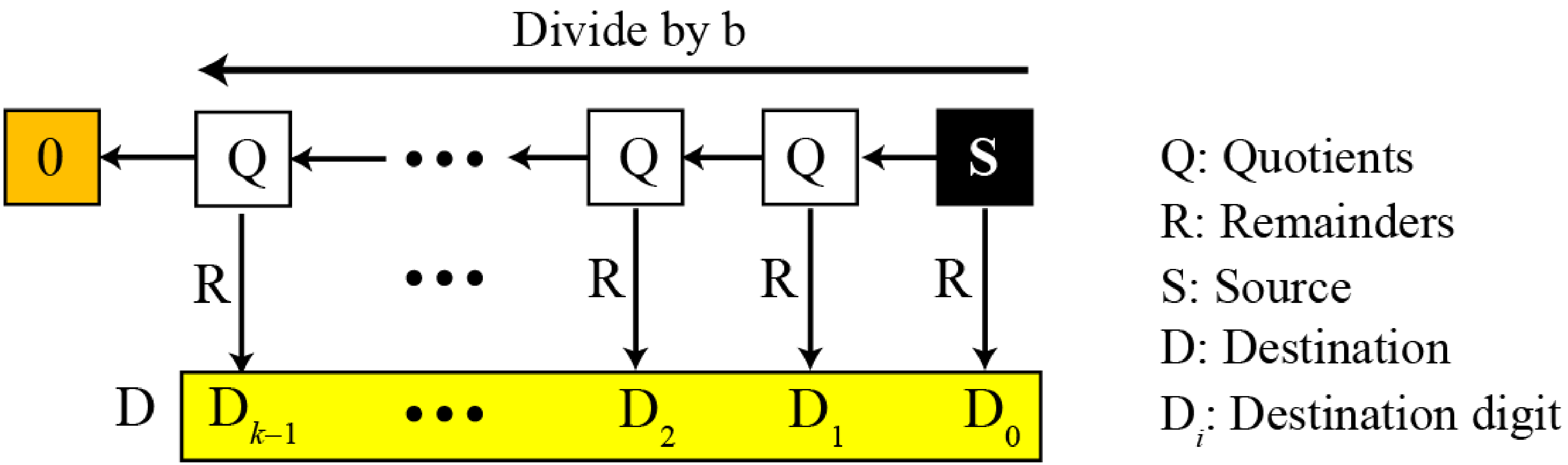
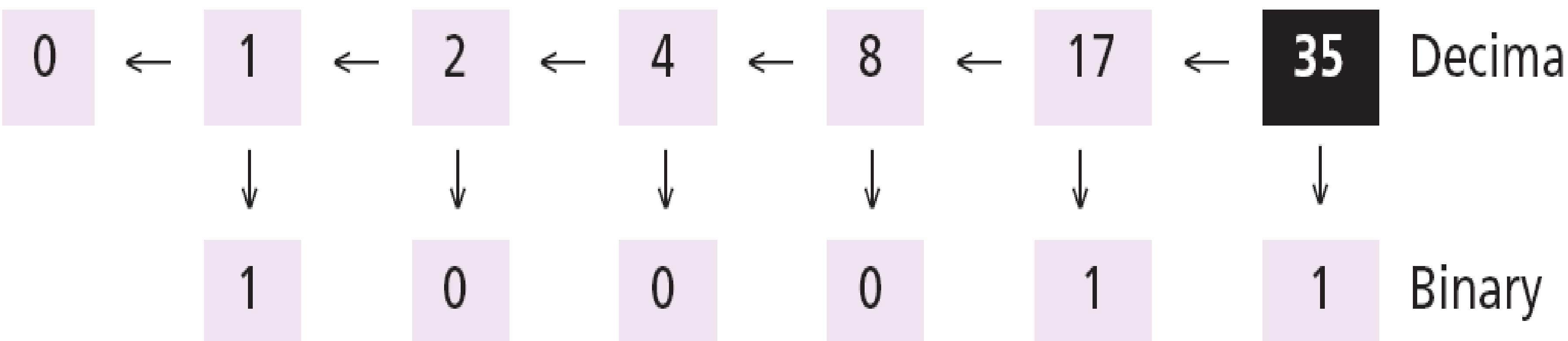


Figure 2.7 **Converting the integral part of a number in decimal to other bases**

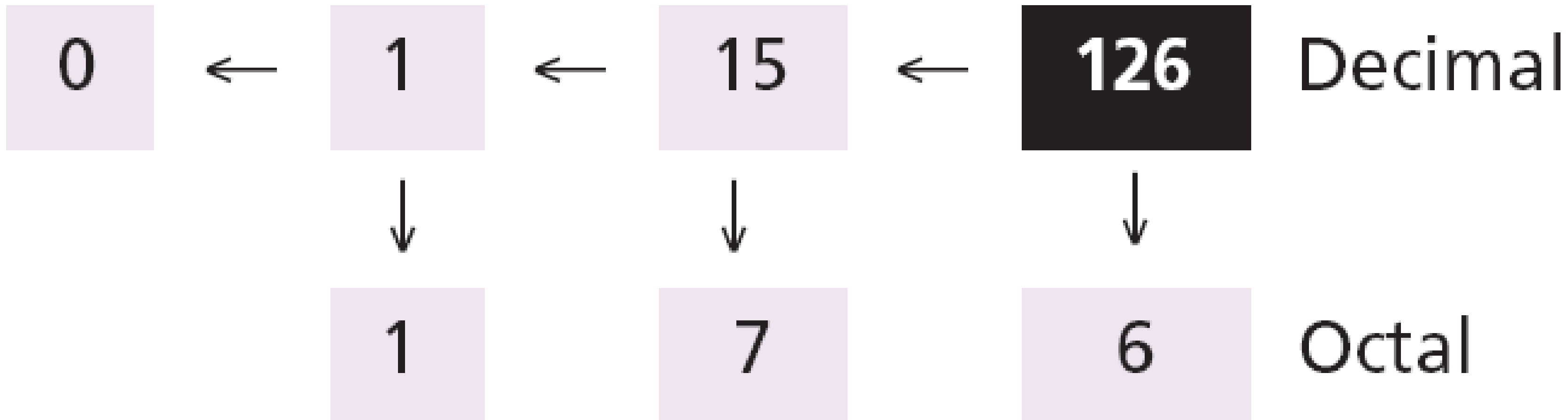
Example 2.11

The following shows how to convert **35** in decimal to binary. We start with the number in decimal, we move to the left while continuously finding the quotients and the remainder of division by 2. The result is $35 = (100011)_2$.



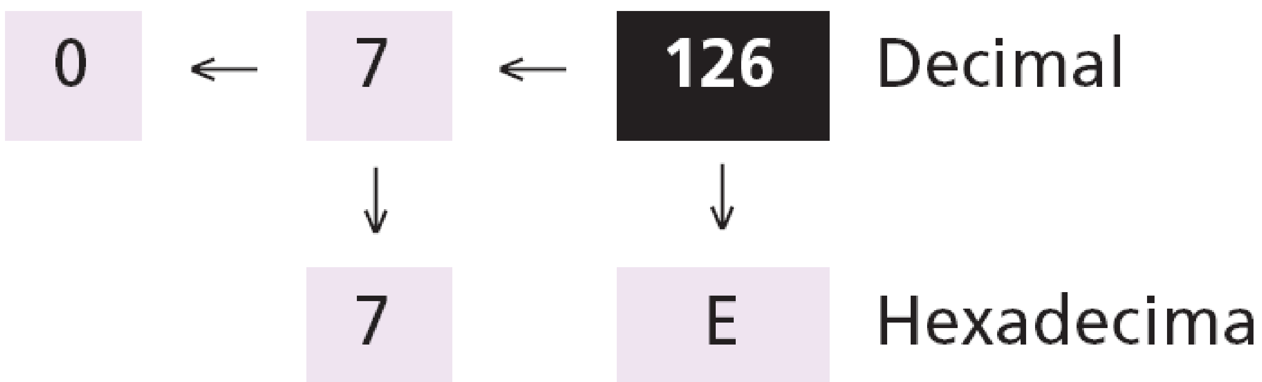
Example 2.12

The following shows how to convert 126 in decimal to its equivalent in the octal system. We move to the right while continuously finding the quotients and the remainder of division by 8. The result is $126 = (176)_8$.



Example 2.13

The following shows how we convert 126 in decimal to its equivalent in the hexadecimal system. We move to the right while continuously finding the quotients and the remainder of division by 16. The result is $126 = (7E)_{16}$



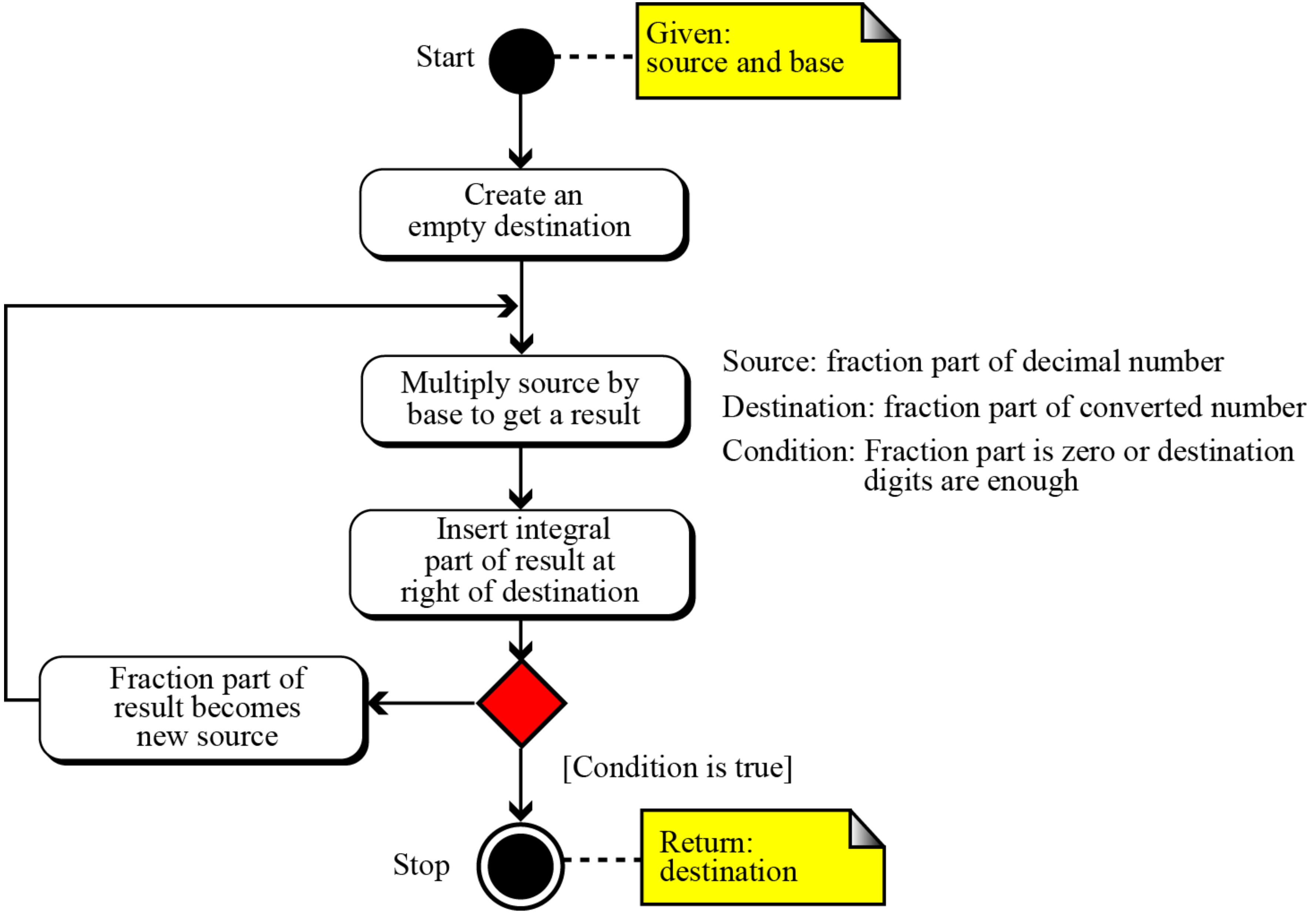


Figure 2.8 **Converting the fractional part of a number in decimal to other bases**

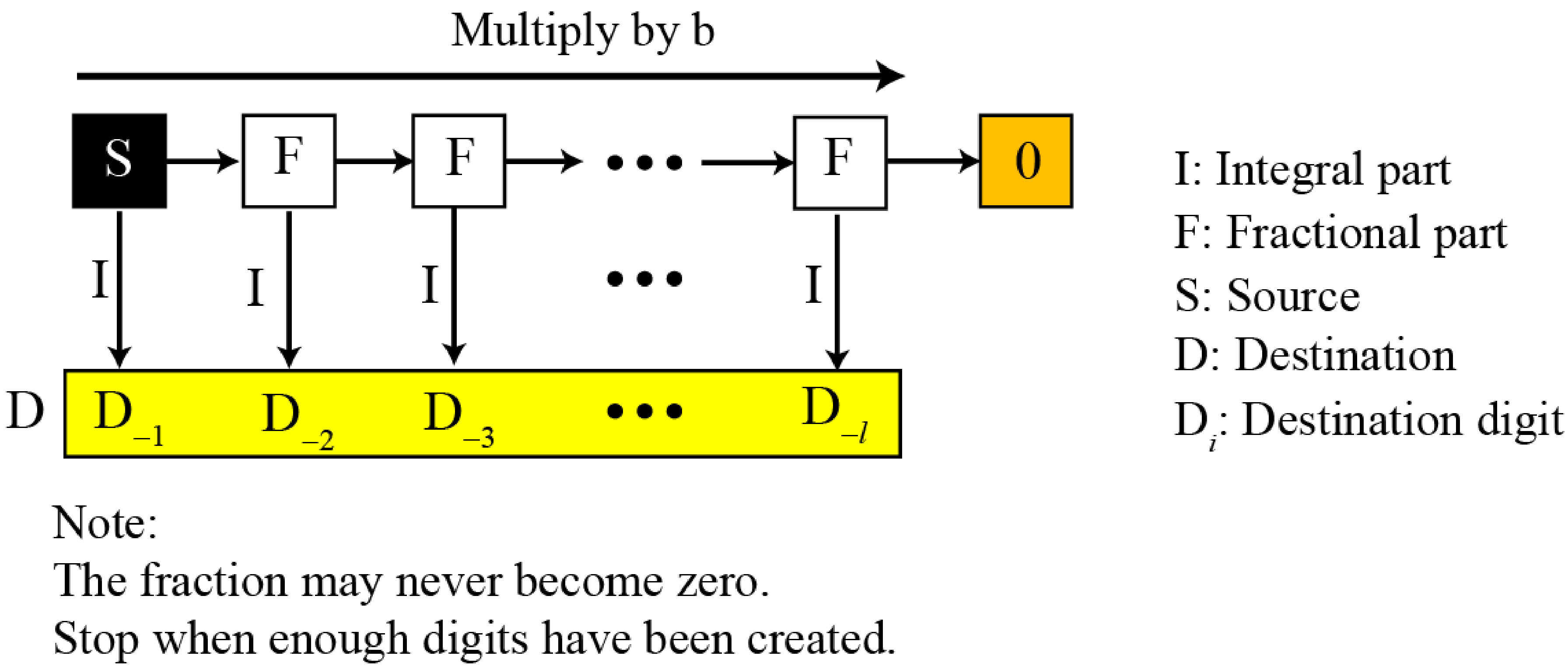
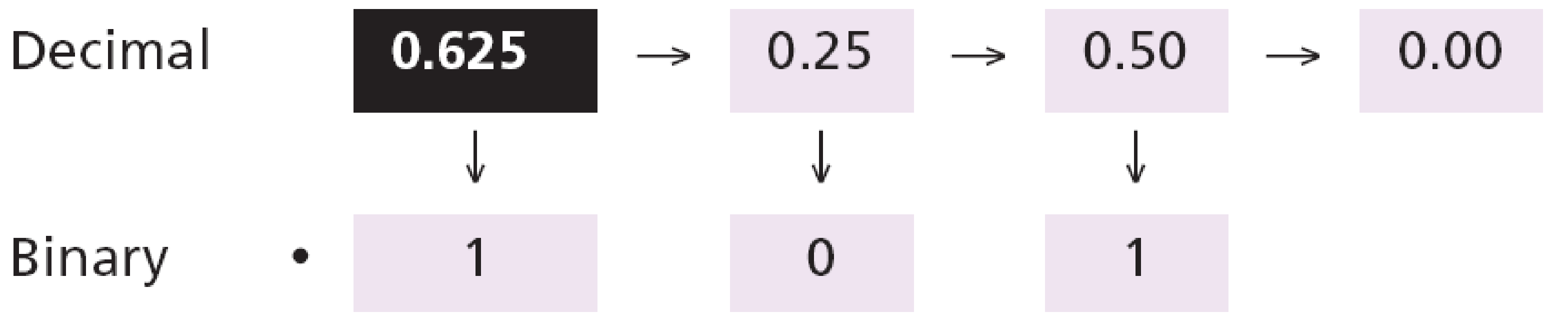


Figure 2.9 **Converting the fractional part of a number in decimal to other bases**

Example 2.14

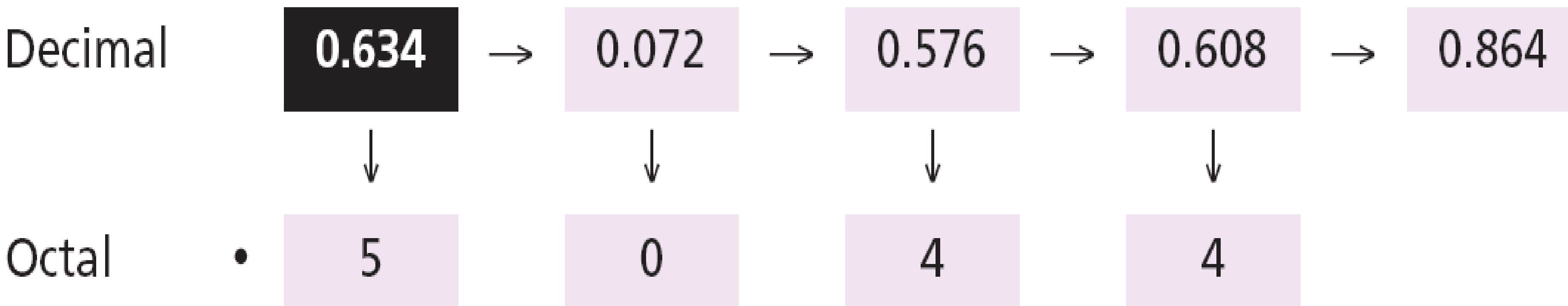
Convert the decimal number 0.625 to binary.



Since the number $0.625 = (0.101)_2$ has no integral part, the example shows how the fractional part is calculated.

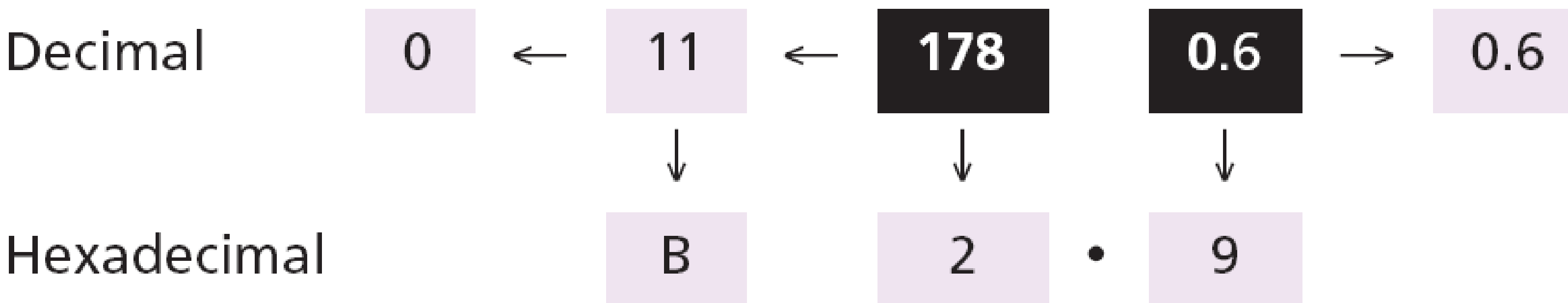
Example 2.15

The following shows how to convert 0.634 to octal using a maximum of four digits. The result is $0.634 = (0.5044)_8$. Note that we multiple by 8 (base octal).



Example 2.16

The following shows how to convert 178.6 in decimal to hexadecimal using only one digit to the right of the decimal point. The result is $178.6 = (B2.9)_{16}$ Note that we divide or multiple by 16 (base hexadecimal).



Example 2.17

An alternative method for converting a small decimal integer (usually less than 256) to binary is to break the number as the sum of numbers that are equivalent to the binary place values shown:

Place values	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Decimal equivalent	128	64	32	16	8	4	2	1

Decimal 165 =	128	+	0	+	32	+	0	+	0	+	4	+	0	+	1
Binary	1		0		1		0		0		1		0		1

Example 2.18

A similar method can be used to convert a decimal fraction to binary when the denominator is a power of two:

Place values	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	2^{-6}	2^{-7}
Decimal equivalent	1/2	1/4	1/8	1/16	1/32	1/64	1/128

Decimal = 27/64	16/64	+	8/64	+	2/64	+	1/64
	1/4	+	1/8	+	1/32	+	1/64

Decimal 27/64 =	0	+	1/4	+	1/8	+	0	+	1/32	+	1/64
Binary	0		1		1		0		1		1

The answer is then $(0.011011)_2$

Binary-hexadecimal conversion

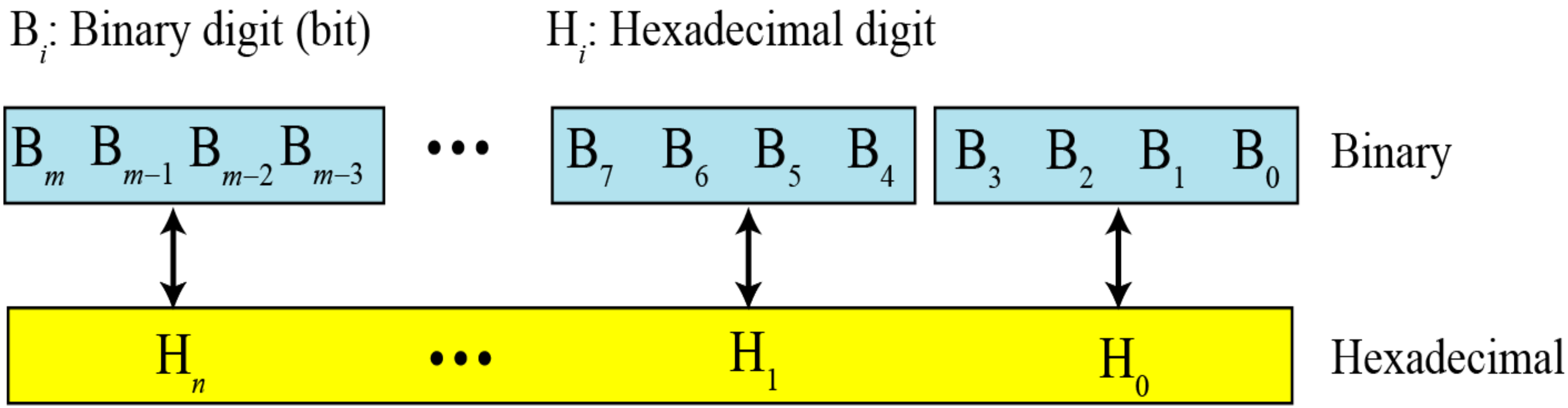


Figure 2.10 **Binary to hexadecimal and hexadecimal to binary conversion**

Example 2.19

Show the hexadecimal equivalent of the binary number $(110011100010)_2$.

Solution

We first arrange the binary number in 4-bit patterns:

1100 1110 0010

Note that the leftmost pattern can have one to four bits. We then use the equivalent of each pattern shown in Table 2.2 on page 25 to change the number to hexadecimal: $(CE2)_{16}$.

Example 2.20

What is the binary equivalent of $(24C)_{16}$?

Solution

Each hexadecimal digit is converted to 4-bit patterns:

$$\mathbf{2 \rightarrow 0010,\ 4 \rightarrow 0100,\ \text{and}\ C \rightarrow 1100}$$

The result is $(001001001100)_2$.

Binary-octal conversion

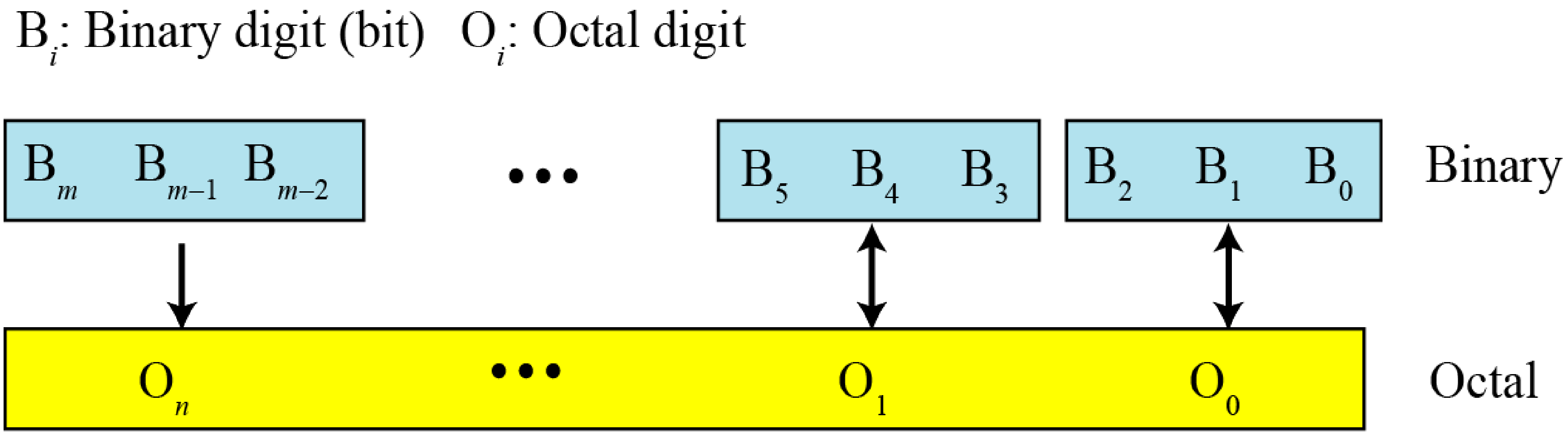


Figure 2.10 **Binary to octal and octal to binary conversion**

Example 2.21

Show the octal equivalent of the binary number $(101110010)_2$.

Solution

Each group of three bits is translated into one octal digit. The equivalent of each 3-bit group is shown in Table 2.2 on page 25.

101 110 010

The result is $(562)_8$.

Example 2.22

What is the binary equivalent of for $(24)_8$?

Solution

Write each octal digit as its equivalent bit pattern to get

$$\mathbf{2 \rightarrow 010 \quad and \quad 4 \rightarrow 100}$$

The result is $(010100)_2$.

Octal-hexadecimal conversion

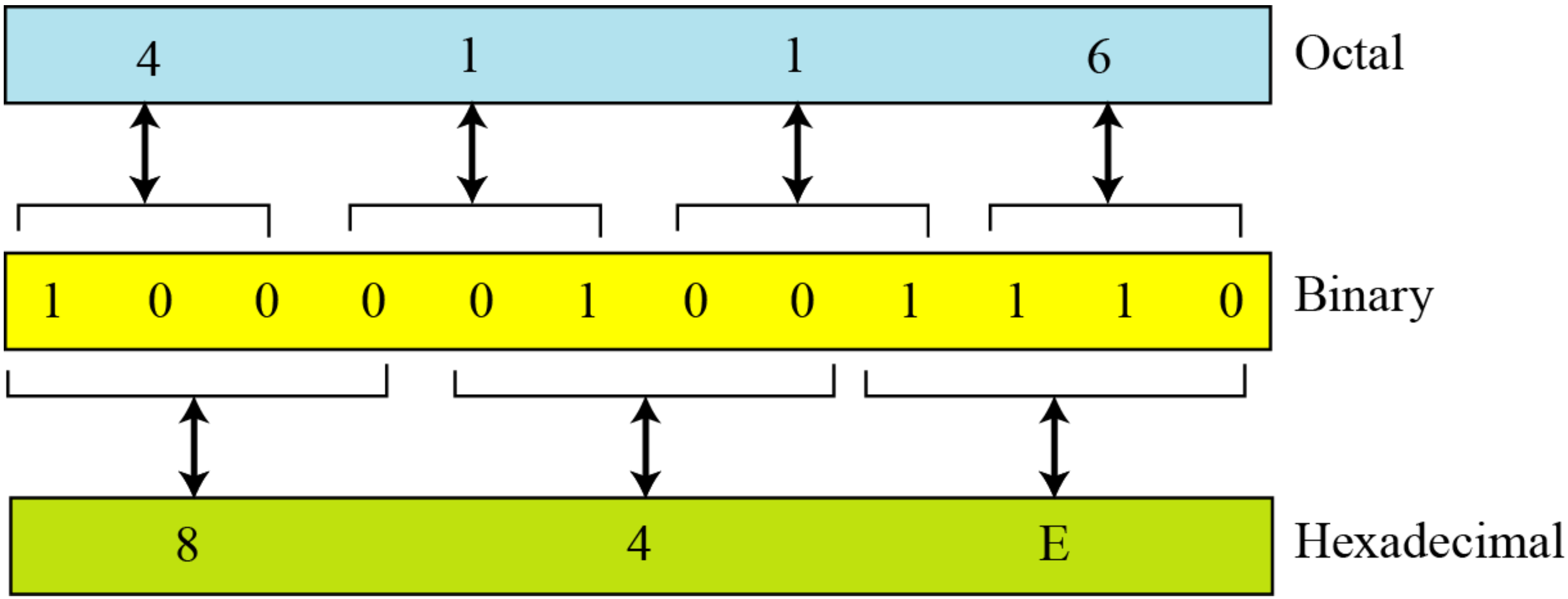


Figure 2.12 **Octal to hexadecimal and hexadecimal to octal conversion**

Example 2.23

Find the minimum number of binary digits required to store decimal integers with a maximum of six digits.

Solution

$k = 6$, $b_1 = 10$, and $b_2 = 2$. Then

$$x = \lceil k \times (\log b_1 / \log b_2) \rceil = \lceil 6 \times (1 / 0.30103) \rceil = 20.$$

The largest six-digit decimal number is 999,999 and the largest 20-bit binary number is 1,048,575. Note that the largest number that can be represented by a 19-bit number is 524,287, which is smaller than 999,999. We definitely need **twenty** bits.

2-3 NONPOSITIONAL NUMBER SYSTEMS

Although **non-positional number systems** are not used in computers, we give a short review here for comparison with positional number systems.

A non-positional number system still uses a limited number of symbols in which each symbol has a value.

In this system, a number is represented as:

$$\boxed{S_{k-1} \ldots S_2\, S_1\, S_0 \bullet S_{-1}\, S_{-2} \ldots S_{-l}}$$

and has the value of:

$$n = \pm \underbrace{S_{k-1} + \ldots + S_1 + S_0}_{\textit{Integral part}} + \underbrace{S_{-1} + S_{-2} + \ldots + S_{-l}}_{\textit{Fractional part}}$$

There are some exceptions to the addition rule we just mentioned, as shown in Example 2.24.

Example 2.24

Roman numerals are a good example of a non-positional number system. This number system has a set of symbols S = {I, V, X, L, C, D, M}. The values of each symbol are shown in Table 2.3

Table 2.3 Values of symbols in the Roman number system

Symbol	*I*	*V*	*X*	*L*	*C*	*D*	*M*
Value	1	5	10	50	100	500	1000

To find the value of a number, we need to add the value of symbols subject to specific rules (See the textbook).

Example 2.24 (Continued)

The following shows some Roman numbers and their values.

III	$\rightarrow$	$1 + 1 + 1$	=	3
IV	$\rightarrow$	$5 - 1$	=	4
VIII	$\rightarrow$	$5 + 1 + 1 + 1$	=	8
XVIII	$\rightarrow$	$10 + 5 + 1 + 1 + 1$	=	18
XIX	$\rightarrow$	$10 + (10 - 1)$	=	19
LXXII	$\rightarrow$	$50 + 10 + 10 + 1 + 1$	=	72
CI	$\rightarrow$	$100 + 1$	=	101
MMVII	$\rightarrow$	$1000 + 1000 + 5 + 1 + 1$	=	2007
MDC	$\rightarrow$	$1000 + 500 + 100$	=	1600