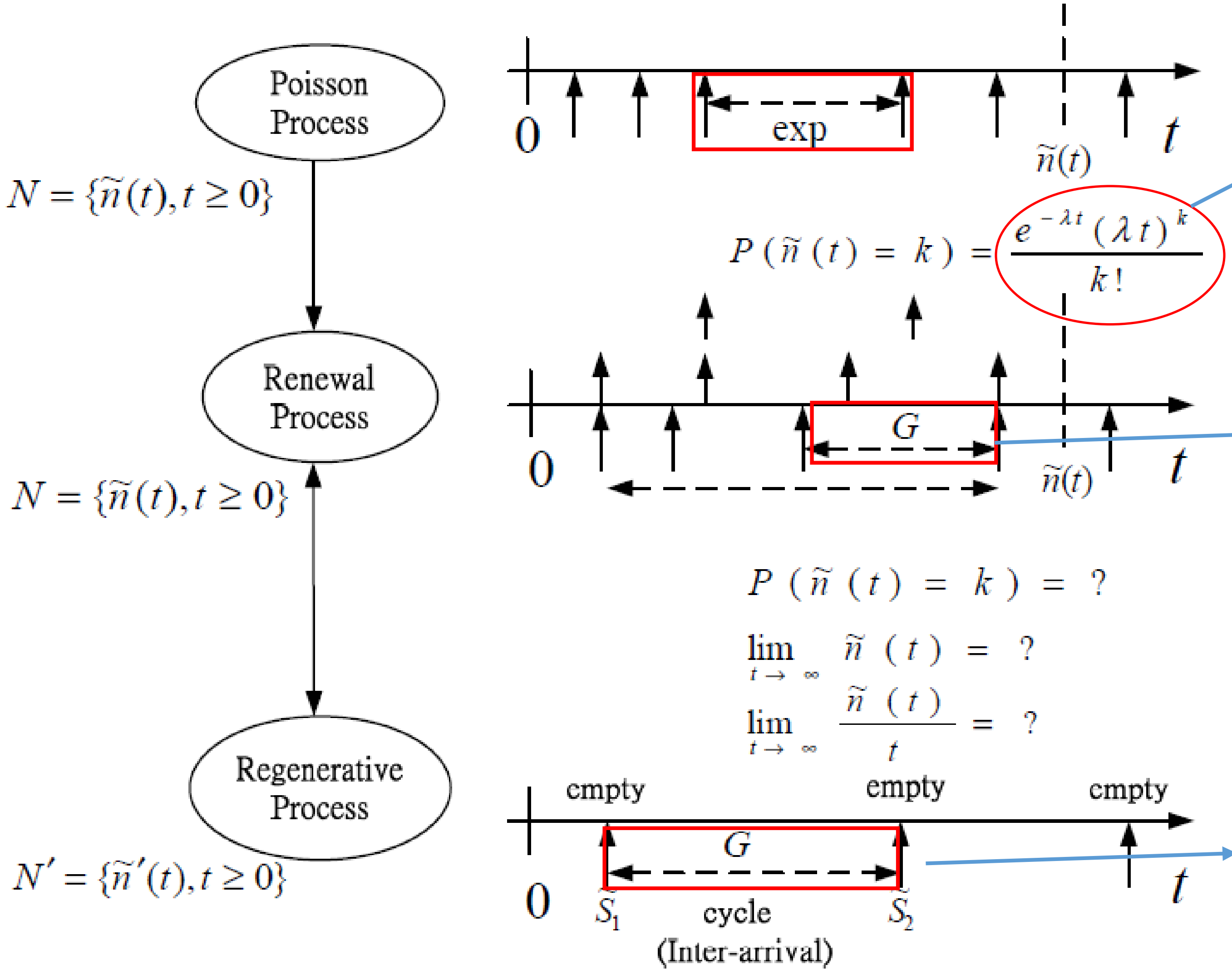
Chapter 6 Renewal Processes

Renewal Processes



(embedded renewal points form a new Renewal process)

在任意一段時間觀察的**distribution,**
是**Possion distribution**

是**General distribution,** 但**arrival**彼此是**i.i.d**

從每一個**arrival** 中,
挑出**process**與**process** 間是**i.i.d**

Outline

在Possion process中是,
Possion distribution,
但在renewal process則是
General distribution

- Distribution and Limiting Behavior of $\tilde{n}(t)$
  - Pmf of $\tilde{n}(t)$: $P(\tilde{n}(t) = k)$ =?
  - Limiting time average : $\lim_{t\to\infty} \frac{\tilde{n}(t)}{t}$ =? (Law of Large Numbers)
  - Limiting PDF of $\tilde{n}(t)$ (Central Limit Theorem)
- Renewal Function $E[\tilde{n}(t)]$, and its Asymptotic (Limiting) behavior
  - Renewal Equation
  - Wald's Theorem and Stopping time
  - Elementary Renewal Theorem
  - Blackwell's Theorem

probability mass function (pmf)

probability density function (pdf)

Outline

- Key Renewal Theorem and Applications
    - Definition of Regenerative Process
    - Renewal Theory
    - Key Renewal Theorem
    - Application 1: Residual Life, Age, and Total Life
    - Application 2: Alternating Renewal Process/Theory
    - Application 3: Mean Residual Life
- Renewal Reward Processes and Applications
    - Renewal Reward Process/Theory
    - Application 1: Alternating Renewal Process/Theory
    - Application 2: Time Average of Residual Life and Age
- More Notes on Regenerative Processes

Introduction

- A **renewal process** is a generalization of the Poisson process.
- In essence, the Poisson process is a continuous-time Markov process on the positive integers (usually starting at zero) which has independent identically distributed *holding times* at each integer *i* (exponentially distributed) before advancing (with probability 1) to the next integer *i+1.*
- In the same informal spirit, we may define a renewal process to be the same thing, except that the *holding times* take on a more general distribution.
- Note that the independence and identical distribution (IID) property of the holding times is retained.

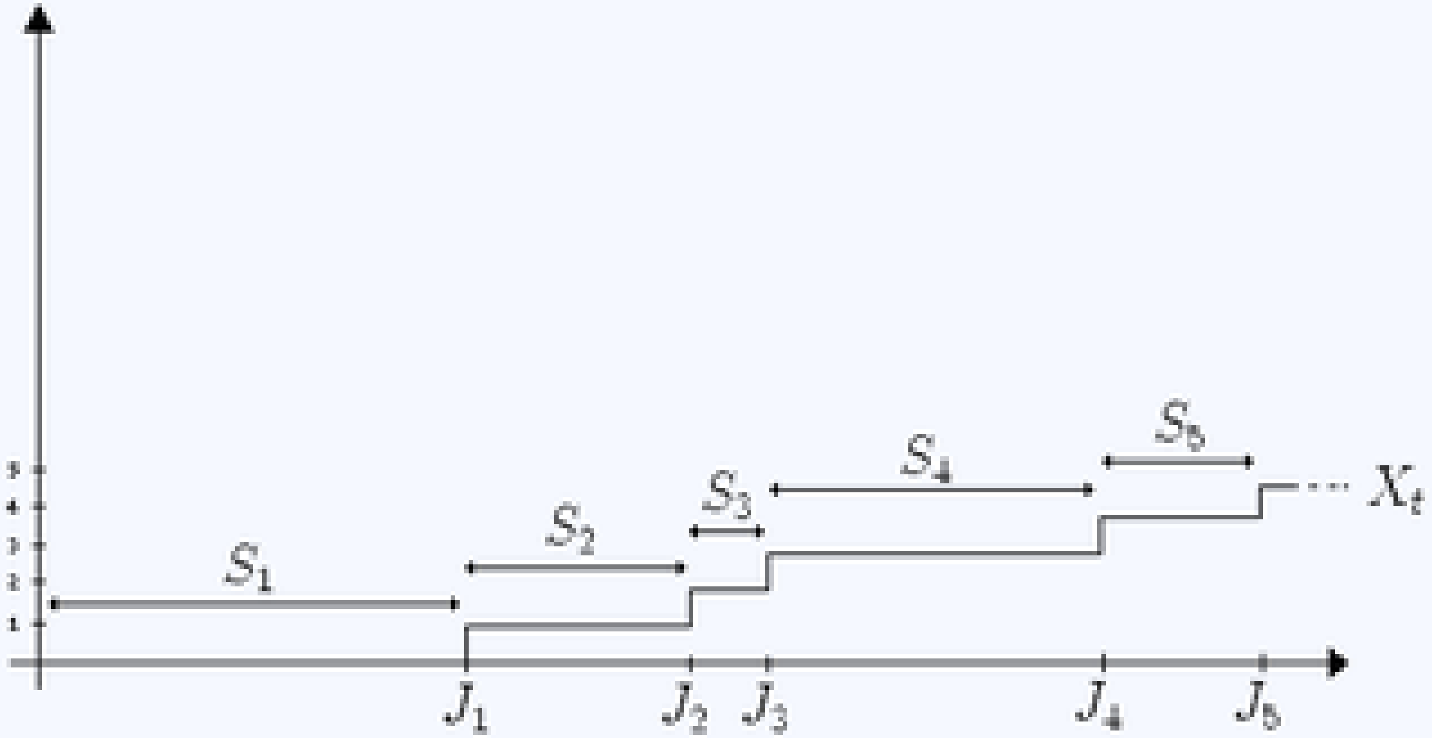
Introduction

Formal definition

- Let S_1 , S_2 , S_3 , S_4 , S_5, be a sequence of positive independent identically distributed random variables such that

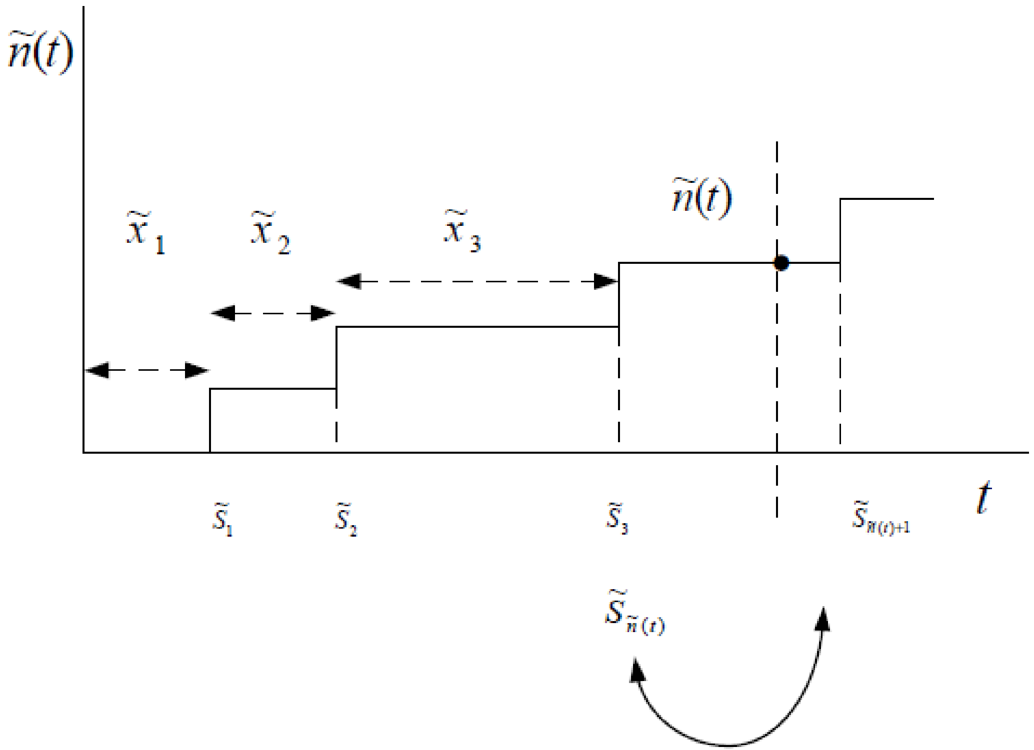
$$\mathbf{0} < \boldsymbol{E}[\mathbf{S_i}] < \infty$$

- We refer to the random variable $\mathbf{S_i}$ as the "*i-th*" holding time and $\boldsymbol{E}[\mathbf{S_i}]$ is the expectation of $\mathbf{S_i}$
- Define for each *n* > 0 : $J_n = \sum_{i=1}^{n} S_i$
- Each J_n is referred to as the "*n-th*" *jump time* and the intervals $[Jn, Jn_{+1}]$ being called *renewal intervals*.



[*renewal interval*]

Distribution and Limiting Behavior of $\tilde{n}(t)$



$\{\tilde{x}_n, n = 1, 2, \ldots\} \sim F_{\tilde{x}}$; mean $\bar{X}$ $(0 < \bar{X} < \infty)$

$N = \{\tilde{n}(t), t \geq 0\}$ is called a renewal (counting) process

$$\tilde{n}(t) = \max\{n : \tilde{S}_n \leq t\}$$

Distribution and Limiting Behavior of $\tilde{n}(t)$

- $\tilde{n}(t)$

1. pmf of $\tilde{n}(t)$→ closed-form
2. Limiting time average [Law of Large Numbers]:

with probability 1

$$\frac{\tilde{n}(t)}{t} \overset{w.p.1}{\longrightarrow} \frac{1}{\bar{X}}, \; t \to \infty$$

3. Limiting time and ensemble average

[Elementary Renewal Theorem]:

$$\frac{E[\tilde{n}(t)]}{t} \overset{w.p.1}{\longrightarrow} \frac{1}{\bar{X}}, \; t \to \infty$$

Distribution and Limiting Behavior of $\tilde{n}(t)$

4. Limiting ensemble average (focusing on arrivals in the vicinity of t)
   [Blackwell's Theorem]:

$$\frac{E[\tilde{n}(t+\delta)-\tilde{n}(t)]}{\delta} \overset{w.p.1}{\rightarrow} \frac{1}{\bar{X}}\ ,\ t \rightarrow \infty$$

5. Limiting PDF of $\tilde{n}(t)$ [Central Limit Theorem]:

$$\lim_{t\rightarrow\infty} P\left[\frac{\tilde{n}(t)-t/\bar{X}}{\sigma\sqrt{t}(\bar{X})^{-3/2}} < y\right] = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \sim Gaussian(\frac{t}{\bar{X}}, \sigma\sqrt{t}\cdot\bar{X}^{-\frac{3}{2}})$$

pmf of $\tilde{n}(t)$

在時間t, 第n個人來的機率;

表示在t時間之前, 人數一定小於等於n

$$
\begin{aligned}
P[\tilde{n}(t) = n] &= P[\tilde{n}(t) \geq n] - P[\tilde{n}(t) \geq n+1] \\
&= P[\tilde{S}_n \leq t] - P[\tilde{S}_{n+1} \leq t] \\
&\qquad \because \tilde{x}_i \sim F, \\
&\qquad \therefore \sum \tilde{x}_i \sim F(t) \otimes F(t) \ldots \otimes F(t) \equiv F_n(t) \\
&= F_n(t) - F_{n+1}(t) \qquad n\text{-fold convolution of } F(t)
\end{aligned}
$$

n 次折積

Limiting Time Average

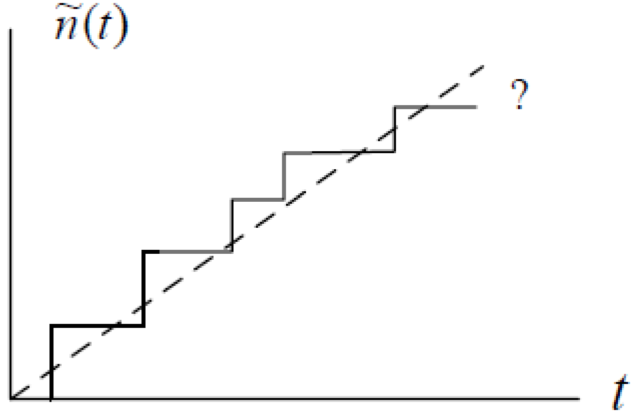
- $\lim_{t\to\infty} \tilde{n}(t)$ =?

$$\because P\left[\lim_{t\to\infty} \tilde{n}(t) < \infty\right] = P\left[\tilde{n}(\infty) < \infty\right] = P\left[\tilde{x}_n = \infty \text{ for some } n\right]$$

$$= P\left[\bigcup_{n=1}^{\infty} (\tilde{x}_n = \infty)\right] = \sum_{n=1}^{\infty} P\left[\tilde{x}_n = \infty\right] = 0$$

$$\therefore \lim_{t\to\infty} \tilde{n}(t) = \tilde{n}(\infty) = \infty \qquad w.p.1$$

- Question: What is the rate at which $\tilde{n}(t)$ goes to ∞ ?



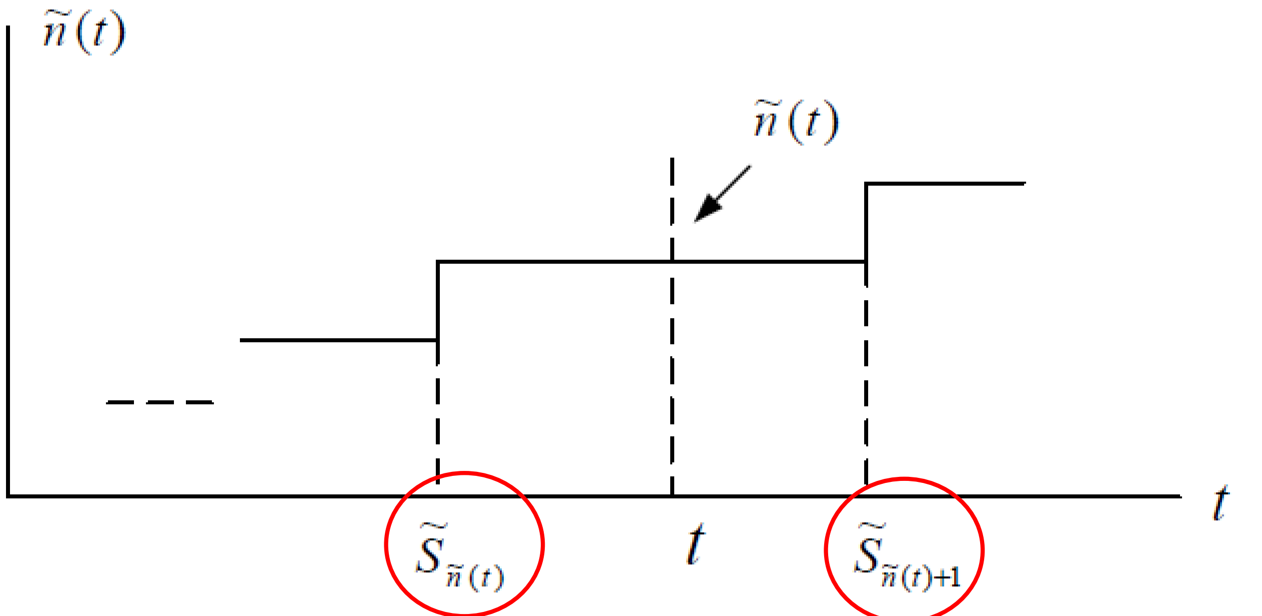
i.e. $\lim_{t\to\infty} \frac{\tilde{n}(t)}{t} = ?$

Strong Law for Renewal Processes

- **Theorem.** For a renewal process $N = \{\tilde{n}(t), t \geq 0\}$ with mean inter-renewal interval $\bar{X}$, then

$$\lim_{t \to \infty} \frac{\tilde{n}(t)}{t} = \frac{1}{\bar{X}}, \quad w.p.1$$

- Proof:



Strong Law for Renewal Processes

同乘$\tilde{n}(t)+1$

$$\because \quad \tilde{S}_{\tilde{n}(t)} \le t < \tilde{S}_{\tilde{n}(t)+1}$$ → 同除$\tilde{n}(t)$

$$\Rightarrow \frac{\tilde{S}_{\tilde{n}(t)}}{\tilde{n}(t)} \le \frac{t}{\tilde{n}(t)} < \frac{\tilde{S}_{\tilde{n}(t)+1}}{\tilde{n}(t)} = \frac{\tilde{S}_{\tilde{n}(t)+1}}{\tilde{n}(t)+1} \times \frac{\tilde{n}(t)+1}{\tilde{n}(t)}$$

Squeeze Theorem

$$\Rightarrow \underbrace{\lim_{t\to\infty} \frac{\tilde{S}_{\tilde{n}(t)}}{\tilde{n}(t)}}_{=\bar{X} \text{ why?}} \le \lim_{t\to\infty} \frac{t}{\tilde{n}(t)} < \lim_{t\to\infty} \left[\underbrace{\frac{\tilde{S}_{\tilde{n}(t)+1}}{\tilde{n}(t)+1}}_{=\bar{X}} \times \underbrace{\frac{\tilde{n}(t)+1}{\tilde{n}(t)}}_{=1} \right]$$

$$\therefore \quad \lim_{t\to\infty} \frac{\tilde{n}(t)}{t} = \frac{1}{\bar{X}}$$

strong law of large number

Renewal Function $E[\tilde{n}(t)]$

- Let $m(t) = E[\tilde{n}(t)]$, which is called "*renewal function*".

1. Relationship between $m(t)$ and F_n

$$m(t) = \sum_{n=1}^{\infty} F_n(t), \qquad \text{where } F_n \text{ is the } n\text{-fold convolution of } F$$

2. Relationship between $m(t)$ and F

   [Renewal Equation]

$$m(t) = F(t) + \int_0^t m(t-x)dF(x)$$

3. Relationship between $m(t)$ and $L_{\tilde{x}}(r)$ (Laplace Transform of $\tilde{x}$)

$$L_m(r) = \frac{L_{\tilde{x}}(r)}{r[1 - L_{\tilde{x}}(r)]}$$

Renewal Function $E[\tilde{n}(t)]$

→ [Wald's Equation]

4. Asymptotic behavior of $m(t)$ ($t \to \infty$, Limiting)
   → [Elementary Renewal Theorem]
   → [Blackwell's Theorem]

Renewal Function $E[\tilde{n}(t)]$

1. $m(t) = E[\tilde{n}(t)] \overset{?}{\longleftrightarrow} F_n$ (i.e., PDF of $\tilde{S}_n$)

$$\text{Let } \tilde{n}(t) = \sum_{n=1}^{\infty} I_n, \text{ where } I_n = \begin{cases} 1, & n_{th} \text{ renewal occurs in } [0,t]; \\ 0, & \text{Otherwise}; \end{cases}$$

$$\begin{aligned} m(t) = E[\tilde{n}(t)] &= E\left[\sum_{n=1}^{\infty} I_n\right] \\ &= \sum_{n=1}^{\infty} E[I_n] \\ &= \sum_{n=1}^{\infty} P[n_{th} \text{ renewal occurs in } [0,t]] \\ &= \sum_{n=1}^{\infty} P[\tilde{S}_n \le t] \end{aligned}$$

Indicator Function

Renewal Function $E[\tilde{n}(t)]$

$$\therefore m(t) = \sum_{n=1}^{\infty} F_n(t)$$ → Convolution Function

or $$m(t) = \sum_{n=1}^{\infty} P[\tilde{n}(t) \geq n] = \sum_{n=1}^{\infty} P[\tilde{S}_n \leq t] = \sum_{n=1}^{\infty} F_n(t)$$

For any non-negative random variable $\tilde{x}$ $$E[\tilde{x}] = \sum_{k=0}^{\infty} p(\tilde{x} > k)$$ discrete

. . . .

As $t \to \infty$, $n \to \infty$, finding F_n is far too complicated

$\Rightarrow$ find another way of solving $m(t)$ in terms of $F_{\tilde{x}}(t)$

Renewal Function $E[\tilde{n}(t)]$

第n個到第n-1的interval time

2. $m(t) \overset{?}{\longleftrightarrow} F_{\tilde{x}}(t)$ (i.e., PDF of $\tilde{x}$)

Distribution Function

$$\because \quad \tilde{S}_n = \tilde{S}_{n-1} + \tilde{x}_n, \text{ for all } n \geq 1, \text{ and } \tilde{S}_{n-1} \text{ and } \tilde{x}_n \text{ are independent,}$$

$$\therefore \quad P[\tilde{S}_n \leq t] = \int_0^t P[\tilde{S}_{n-1} \leq t - x] dF_{\tilde{x}}(x), \text{ for } n \geq 2$$

第n-1個到第n個的time

$$\text{for } n = 1, \tilde{x}_1 = \tilde{S}_1, P[\tilde{S}_1 \leq t] = F_{\tilde{x}}(x)$$

$$\therefore \quad m(t) = \sum_{n=1}^{\infty} P[\tilde{S}_n \leq t] = F_{\tilde{x}}(t) + \int_0^t \sum_{n=2}^{\infty} P[\tilde{S}_{n-1} \leq t - x] dF_{\tilde{x}}(x)$$

$$m(t) = F_{\tilde{x}}(t) + \int_0^t m(t-x) \cdot dF_{\tilde{x}}(x) \quad \Rightarrow \text{Renewal Equation}$$

Renewal Function $E[\tilde{n}(t)]$

3. $L_m(r) \overset{?}{\longleftrightarrow} L_{\tilde{x}}(r)$ (Laplace Transform of $\tilde{x}$)
(Laplace Transform of $m(t) = L_m(r)$)

Answer:

$$L_m(r) = \frac{L_{\tilde{x}}(r)}{r[1 - L_{\tilde{x}}(r)]}$$

<Homework> Prove it.

4. Asymptotic behavior of $m(t)$:

$$\lim_{t\to\infty} \frac{m(t)}{t} = \lim_{t\to\infty} \frac{E[\tilde{n}(t)]}{t} = ?$$

Stopping time

- In probability theory, in particular in the study of stochastic processes, a **stopping time** (also **Markov time**) is a specific type of “random time”: a random variable whose value is interpreted as the time at which a given stochastic process exhibits a certain behavior of interest.

- A stopping time is often defined by a **stopping rule**, a mechanism for deciding whether to continue or stop a process on the basis of the present position and past events, and which will almost always lead to a decision to stop at some finite time.

Stopping Time (Rule)

$\tilde{N}$只要觀察$\tilde{x}_1, \tilde{x}_2, \ldots \tilde{x}_n$就可以，
不用考慮$\tilde{x}_{n+1}, \tilde{x}_{n+2}, \ldots$，
就是一個stopping time

Definition. $\tilde{N}$, an integer-valued r.v., is said to be a "*stopping time*" for a set of independent random variables $\tilde{x}_1, \tilde{x}_2, \ldots$ if event $\{\tilde{N} = n\}$ is independent of $\tilde{x}_{n+1}, \tilde{x}_{n+2}, \ldots$

Example 1.

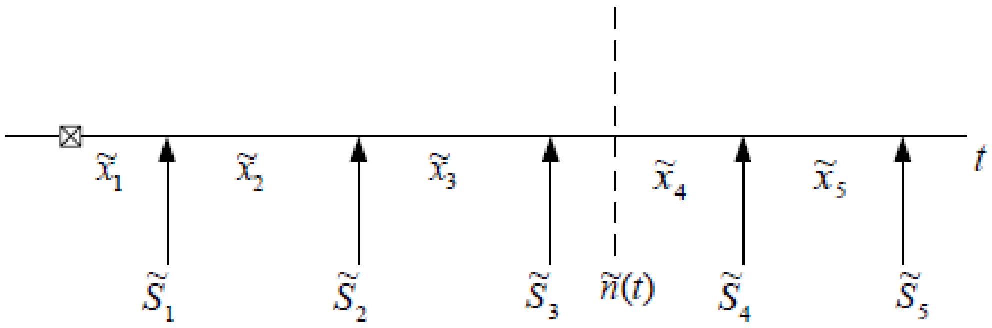
- Let $\tilde{x}_1, \tilde{x}_2, \ldots$ be independent random variables,
- $P[\tilde{x}_n = 0] = P[\tilde{x}_n = 1] = 1/2$, $n = 1, 2, \ldots$
- if $\tilde{N} = \min\{n : \tilde{x}_1 + \ldots + \tilde{x}_n = 10\}$

→ Is $\tilde{N}$ a stopping time for $\tilde{x}_1, \tilde{x}_2, \ldots$? **Yes**

Stopping Time (Rule)

Example 2.

- $\tilde{n}(t)$, $X = \{\tilde{x}_n, n = 1, 2, 3, \ldots\}$,
- $S = \{\tilde{S}_n, n = 0, 1, 2, 3, \ldots\}$,
- $\tilde{S}_n = \tilde{S}_{n-1} + \tilde{x}_n$



→ Is $\tilde{n}(t)$ the stopping time of $X = \{\tilde{x}_n, n = 1, 2, \ldots\}$? No

Stopping Time (Rule)

Example 3. Is $\tilde{n}(t)$+ 1 the stopping time for $\{\tilde{x}_n\}$? **Yes**

Answer:

跟S_{n+1}無關, 只跟S_{n-1}到 S_n有關

$$\text{whether } \tilde{n}(t)+1=n\ (\rightarrow \tilde{n}(t)=n-1) \text{ depends on } \tilde{S}_{n-1}\le t<\tilde{S}_n$$

$$\therefore \text{ depends on } \tilde{S}_{n-1} \text{ and } \tilde{S}_n, \text{ i.e., up to } \tilde{x}_n$$

$$\therefore\ \tilde{n}(t)+1 \text{ is the stopping time for } \{\tilde{x}_n\}, \text{ so is } \tilde{n}(t)+2, \tilde{n}(t)+3, \ldots$$

Stopping Time - from $\tilde{I}_n$

- **Definition.** $\tilde{N}$, an integer-valued r.v. is said to be a *stopping time* for a set of independent random variables $\{\tilde{x}_n, n \geq 1\}$, if for each $n > 1$, $\tilde{I}_n$, conditional on $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_{n-1}$, is independent of $\{\tilde{x}_k, k \geq n\}$
- **Define.** $\tilde{I}_n$ - a decision rule for stopping time $\tilde{N}$, $n \geq 1$

$$\tilde{I}_n = \begin{cases} 1, & \text{if the } n_{th} \text{ observation is to be made;} \\ 0, & \text{Otherwise} \end{cases}$$

1. $\because$ $\tilde{N}$ is the stopping time
   $\therefore$ $\tilde{I}_n$ depends on $\tilde{x}_1, \ldots, \tilde{x}_{n-1}$ but not $\tilde{x}_n, \tilde{x}_{n+1}, \ldots$
2. $\tilde{I}_n$ is also an indicator function of event $\{\tilde{N} \geq n\}$,
   i.e., $\tilde{I}_n = \begin{cases} 1, & \text{if } \tilde{N} \geq n; \\ 0, & \text{Otherwise;} \end{cases}$

Stopping Time - from $\tilde{I}_n$

Because

If $\widetilde{N} \geq n$, then n_{th} observation must be made;

Since $\widetilde{N} \geq n$ implies $\widetilde{N} \geq n - 1$ and happily, $\tilde{I}_n = 1$ implies $\tilde{I}_{n\,\_\,1} = 1$

$\therefore$ Stopping time

$$\begin{cases} \{\tilde{N} = n\}, \text{ is independent of } \tilde{x}_{n+1}, \tilde{x}_{n+2}, \ldots \\ \qquad \underline{\text{or}} \\ \tilde{I}_n \text{ is independent of } \tilde{x}_n, \tilde{x}_{n+1}, \ldots \end{cases}$$

Wald's Equation

- **Wald's equation**, **Wald's identity** or **Wald's lemma** is an important identity that simplifies the calculation of the expected value of the sum of a random number of random quantities
- It relates the expectation of a sum of randomly many finite-mean, independent and identically distributed random variables to the expected number of terms in the sum and the random variables' common expectation under the condition that the number of terms in the sum is independent of the summands.
- Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of real-valued, independent and identically distributed random variables and let N be a nonnegative integer-value random variable that is independent of the sequence $(X_n)_{n\in\mathbb{N}}$. Suppose that N and the X_n have finite expectations. Then

$$\mathrm{E}[X_1 + \cdots + X_N] = \mathrm{E}[N]\,\mathrm{E}[X_1]$$

Wald's Equation

Theorem. If $\{\tilde{x}_n, n \geq 1\}$ are i.i.d. random variables with finite mean $E[\tilde{x}]$, and if $\tilde{N}$ is the stopping time for $\{\tilde{x}_n, n \geq 1\}$, such that $E[\tilde{N}] < \infty$. Then,

$$E\left[\sum_{n=1}^{\tilde{N}} \tilde{x}_n\right] = E[\tilde{N}] \cdot E[\tilde{x}]$$

Proof. Let $\tilde{I}_n = \begin{cases} 1, & \text{if } n \leq \tilde{N}; \\ 0, & \text{otherwise;} \end{cases}$

當n>$\tilde{N}$ *為 0,*
所以可以寫成$\sum \tilde{x}_n$的式子, 沒有影響

$$E\left[\sum_{n=1}^{\tilde{N}} \tilde{x}_n\right] = E\left[\sum_{n=1}^{\infty} \tilde{x}_n \cdot \tilde{I}_n\right]$$

$$= \sum_{n=1}^{\infty} E[\tilde{x}_n \cdot \tilde{I}_n] = \sum_{n=1}^{\infty} E[\tilde{x}_n] \cdot E[\tilde{I}_n]$$

Wald's Equation

帶入$\tilde{I}_n$的定義

$$= E[\tilde{x}] \sum_{n=1}^{\infty} E[\tilde{I}_n] = E[\tilde{x}] \sum_{n=1}^{\infty} P(\tilde{N} \geq n)$$

$$= E[\tilde{x}] E[\tilde{N}]$$

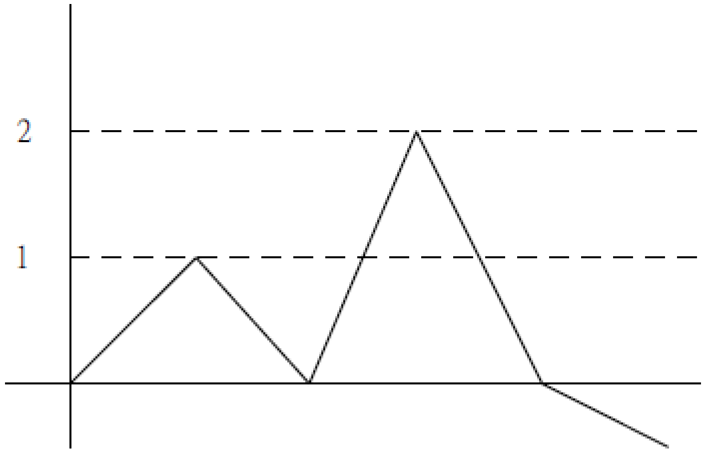
- For Wald's Theorem to be applied, other than *{˜xi, i ≥ 1}*

1. $\tilde{N}$ must be a stopping time; and
2. $E[\tilde{N}] < \infty$

必須要滿足的條件

Wald's Equation

- **Example.** (Example 3.2.3 – Simple Random Walk, [Kao])



$\{\tilde{x}_i\}$ i.i.d. with:

$$P(\tilde{x} = 1) = p$$

$$P(\tilde{x} = -1) = 1 - p = q$$

$$\tilde{S}_n = \sum^n \tilde{x}_k$$

Wald's Equation

$1*p + (-1) * q$

- Let $\widetilde{N} = \min\{n \mid \tilde{S}_n = 1\}$
  $\rightarrow \widetilde{N}$ is the stopping time

$$E[\tilde{S}_n] = E[\widetilde{N}] \cdot E[\tilde{x}] = E[\widetilde{N}] \cdot (p - q)$$

$\because \tilde{S}_{\tilde{N}} = 1 \text{ for all } \tilde{N}$

$\therefore E[\tilde{S}_{\tilde{N}}] = 1$

- if $p = q$, $E[\widetilde{N}] = \infty \Rightarrow$ Wald's Theorem not applicable
- if $p > q$, $E[\widetilde{N}] < \infty \Rightarrow E[\widetilde{N}] = 1/(p-q)$
- if $p < q$, $E[\widetilde{N}] = \infty \Rightarrow$ Wald's Theorem not applicable

$1 = E[\widetilde{N}] \cdot (p - q)$

$E[\widetilde{N}] = 1/(p - q)$

Wald's Equation

- Let $\widetilde{M}$ = min{n| $\tilde{S}_n$= 1} – 1 → $\widetilde{M}$不是stopping time

$$\therefore\ \tilde{S}_{\tilde{M}} = 0 \qquad \rightarrow \qquad E[\tilde{S}_{\tilde{M}}] = 0$$

$$\text{assume } E[\tilde{N}] < \infty,\ p > q,\ \therefore\ E[\tilde{M}] < \infty$$

$$\text{but } \underbrace{E[\tilde{S}_{\tilde{M}}]}_{=0} \neq \underbrace{E[\tilde{M}]}_{\text{finite}}(p-q)$$

Why???

Corollary

- Before proving $\lim_{t\to\infty} \frac{m(t)}{t} \to \frac{1}{\bar{X}}$ *Ref. Page 13*

- **Corollary.** If $\bar{X} < \infty$, then

$$E[\tilde{S}_{\tilde{n}(t)+1}] = \bar{X}[m(t)+1]$$

- **Proof.**

$$E[\tilde{S}_{\tilde{n}(t)+1}] = E\left[\sum_{n=1}^{\tilde{n}(t)+1} \tilde{x}_n\right] = \bar{X} \cdot E[\tilde{n}(t)+1] = \bar{X} \cdot [m(t)+1]$$

Why? ⇨ $Ref.\ page\ 27$

The Elementary Renewal Theorem

- **Theorem.**

$$\frac{m(t)}{t} \to \frac{1}{\bar{X}} \qquad \text{as } t \to \infty$$

- **Proof.**

上限　　　　下限

$$\text{To prove } \underbrace{\frac{1}{\bar{X}} \le \liminf_{t\to\infty} \frac{m(t)}{t}}_{①} \le \underbrace{\limsup_{t\to\infty} \frac{m(t)}{t} \le \frac{1}{\bar{X}}}_{②}$$

$$1.\ \because\ \tilde{S}_{\tilde{n}(t)+1} > t \qquad \therefore \text{ from Cor., } \bar{X}\,[m(t)+1] > t$$

$$\frac{m(t)}{t} \ge \frac{1}{\bar{X}} - \frac{1}{t} \qquad \therefore \liminf_{t\to\infty} \frac{m(t)}{t} \ge \frac{1}{\bar{X}}$$

The Elementary Renewal Theorem

2. Consider <u>a truncated renewal process</u>

最大值為M

$$\tilde{x}'_n = \begin{cases} \tilde{x}_n, & \text{If } \tilde{x}_n \leq M;\ n = 1, 2, \ldots \\ M, & \text{otherwise} \end{cases}$$

Let $\tilde{S}'_n = \sum \tilde{x}'_n$, and $\tilde{N}'(t) = \sup\{n : \tilde{S}'_n \leq t\}$. We have that

$$\tilde{S}'_{\tilde{N}'(t)+1} \leq t + M$$

From the corollary,

$$[m'(t) + 1]\bar{X}' \leq t + M, \quad \text{where } \bar{X}' = E[\tilde{x}'_n]$$

$$\therefore \lim_{t\to\infty} \sup \frac{m'(t)}{t} \leq \frac{1}{\bar{X}'}$$

The Elementary Renewal Theorem

But since $\tilde{S}'_n \leq \tilde{S}_n \quad \rightarrow \quad \tilde{N}'(t) \geq \tilde{N}(t),\ m'(t) \geq m(t)$

$$\therefore \lim_{t\rightarrow\infty} \sup \frac{m(t)}{t} \leq \frac{1}{\bar{X}'}$$

Let $M \rightarrow \infty, \bar{X}' \rightarrow \bar{X}$

$$\therefore \lim_{t\rightarrow\infty} \sup \frac{m(t)}{t} \leq \frac{1}{\bar{X}}$$

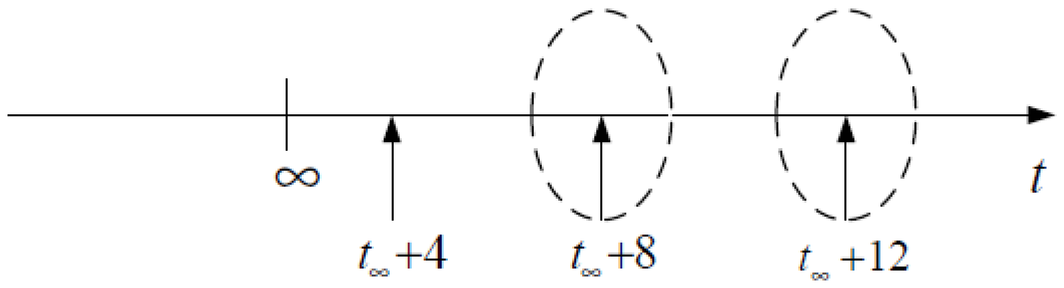
Wald's Equation

Blackwell's Theorem

- **Ensemble Average.**
  - to determine the expected renewal rate in the limit of large t, without averaging from 0 → t (time average)

當t很大時, 觀看在t時間附近的packet arrival情形

- **Question.**
  - are there some values of t at which renewals are more likely than others for large t ?



- **An example.** If each inter-renewal interval $\{\tilde{x}_i, i = 1, 2, . . .\}$ takes on integer number of time units, e.g., 0, 4, 8, 12, . . . , then expected rate of renewals is zero at other times. Such random variable is said to be "*lattice*".

Blackwell's Theorem

Definitions.

- A nonnegative random variable $\tilde{x}$ is said to be *lattice* if there exists $d \geq 0$ such that

$$\sum_{n=0}^{\infty} P[\tilde{x} = nd] = 1$$

- That is, $\tilde{x}$ is lattice if it only takes on integral multiples of some nonnegative number *d*.
- The largest *d* having this property is said to be the *period* of $\tilde{x}$. If $\tilde{x}$ is lattice and *F* is the distribution function of $\tilde{x}$, then we say that *F* is *lattice*.
- **Answer.**
  - Inter-renewal interval random variables are not lattice

$\Rightarrow$ uniform expected rate of renewals in the limit of large *t*.

(*Blackwell's Theorem*)

Blackwell's Theorem

Theorem. If, for $\{\tilde{x}_i, i \geq 1\}$, which are not lattice, then, for any $\delta > 0$,

$$\lim_{t\to\infty} [m(t+\delta) - m(t)] = \frac{\delta}{\bar{X}}$$

Proof. (omitted)

- For non-lattice inter-renewal process $\{\tilde{x}_i, i \geq 1\}$,

1. $\because \tilde{x}_i > 0 \Rightarrow$ No multiple renewals (single arrival)
2. From Blackwell's Theorem, the probability of a renewal in a small interval $(t, t+\delta]$ tends to $\delta/\bar{X} + o(\delta)$ as $t \to \infty$,

$\therefore$ Limiting distribution of renewals in $(t, t+\delta]$ satisfies

$$\lim_{t\to\infty} P[\tilde{n}(t+\delta) - \tilde{n}(t) = 1] = \frac{\delta}{\bar{X}} + o(\delta)$$

Blackwell's Theorem

$$\lim_{t \to \infty} P[\tilde{n}(t+\delta) - \tilde{n}(t) = 0] = 1 - \frac{\delta}{\bar{X}} + o(\delta)$$

$$\lim_{t \to \infty} P[\tilde{n}(t+\delta) - \tilde{n}(t) \geq 2] = o(\delta)$$

Blackwell's Theorem

是指任兩段不重疊的區間
內的事件發生次數互不相干

$\Rightarrow$

	single arrival	Stationary Increment	Independent Increment
Poisson	yes	yes	yes
Renewal Process (Non-lattice)	yes	yes	no

是指某個區間內事件發生
次數的機率分配
只跟那段區間的長度有關。