Basic Queuing Systems

- What is queuing theory?
  - Queuing theory is the study of queues (sometimes called waiting lines)
  - Can be used to describe real world queues, or more abstract queues, found in many branches of computer science, such as operating systems
- Basic queuing theory

  Queuing theory is divided into 3 main sections:
  - Traffic flow
  - Scheduling
  - Facility design and employee allocation

Kendall's Notation

- **D.G. Kendall in 1951 proposed a standard notation for classifying queuing systems into different types.**
- **Accordingly the systems were described by the notation A/B/C/D/E where:**

A	**Distribution of inter arrival times of customers**
B	**Distribution of service times**
C	**Number of servers**
D	**Maximum number of customers in the system**
E	**Calling population size**

Kendall's Notation

David George Kendall

A	顧客到訪間隔之分佈
B	服務時間之分佈
C	伺服器數量
D	系統最大顧客數
E	通話人口大小

Kendall's Notation

A and B can take any of the following distributions types:

M	**Exponential distribution (Markovian)**
D	**Degenerate (or deterministic) distribution**
E_k	**Erlang distribution (*k* = shape parameter)**
H_k	**Hyper exponential with parameter *k***

Kendall's notation

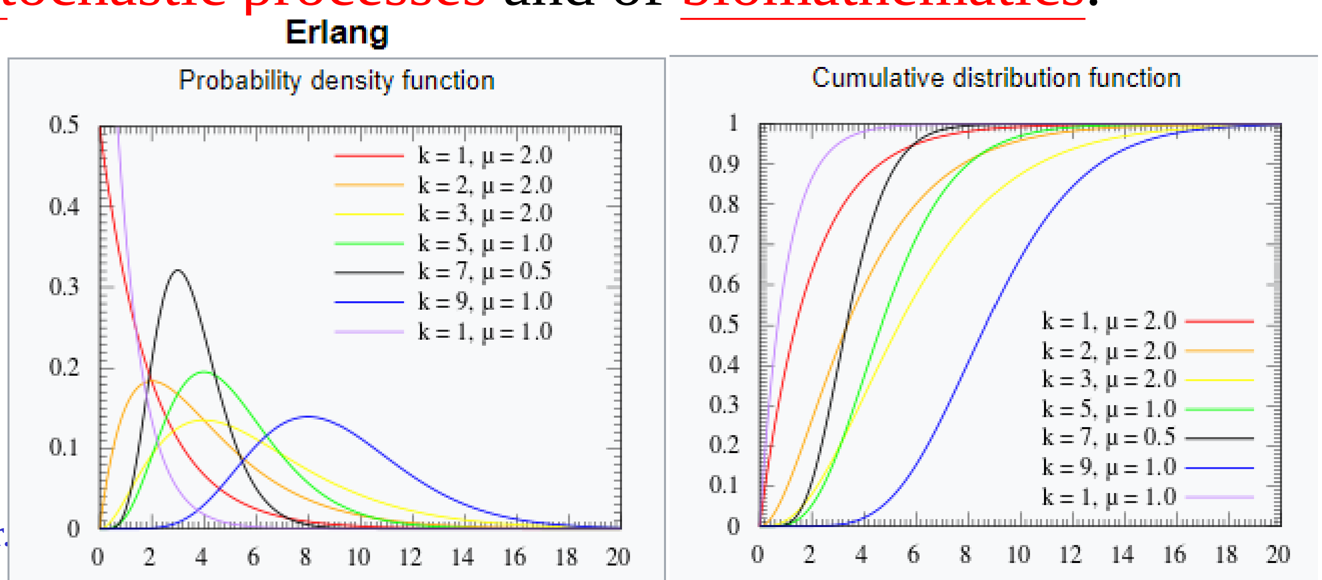
M	指數分佈 (馬可夫: **Markovian**)
D	退化 (**deterministic**) 分佈(或特定性分佈)
E_k	爾朗(**Erlang**)分佈 (***k*** 為外型參數)
G	一般分佈(任意分佈)
H_k	參數為 ***k*** 之超指數

Erlang Distribution

- The Erlang distribution is a two parameter family of continuous probability distributions with support $x \in [0, \infty)$
- The two parameters are:
  - a positive integer *k* the "shape",
  - a positive real number *λ* the "rate".
    - the "scale", *μ,* the reciprocal of the rate, is sometimes used instead.
- The Erlang distribution with shape parameter *k=1* simplifies to the exponential distribution.
  - It is a special case of the Gamma distribution. It is the distribution of a sum *k* independent exponential variables with mean 1/ *λ* each*.*

Erlang Distribution

- The Erlang distribution was developed by A. K. Erlang to examine the number of telephone calls which might be made at the same time to the operators of the switching stations.
- This work on telephone traffic engineering has been expanded to consider waiting times in queueing systems in general. The distribution is now used in the fields of stochastic processes and of biomathematics.



Little's Law

- **Assuming a queuing environment to be operating in a stable steady state where all initial transients have vanished, the key parameters characterizing the system are:**
  - λ – **the mean steady state customer arrival**
  - N – **the average number of customers in the system**
  - T – **the mean time spent by each customer in the system**

which gives

$$N = \lambda T$$

Markov Process

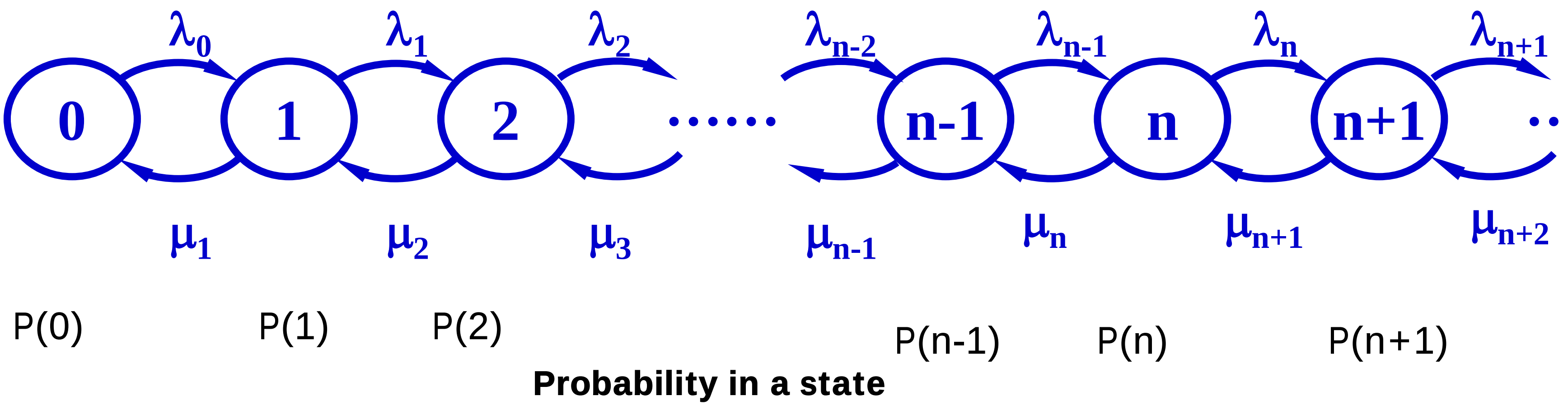
- A Markov process is one in which the next state of the process depends only on the present state, irrespective of any previous states taken by the process
- The knowledge of the current state and the transition probabilities from this state allows us to predict the next state

Russian mathematician Andrey Markov.

Birth-Death Process

- **Special type of Markov process**
- **Often used to model a population (or, no. of jobs in a queue)**
- **If, at some time, the population has *n* entities (*n* jobs in a queue), then birth of another entity (arrival of another job) causes the state to change to $n+1$**
- **On the other hand, a death (a job removed from the queue for service) would cause the state to change to $n-1$**
- **Any state transitions can be made only to one of the two neighboring states**

State Transition Diagram



The state transition diagram of the continuous birth-death process

- 狀態 0 的平衡方程式：$\mu_1 p_1 = \lambda_0 p_0$
- 狀態 1 的平衡方程式：$\lambda_0 p_0 + \mu_2 p_2 = (\lambda_1 + \mu_1) p_1$
- 狀態 n 的平衡方程式：$\lambda_{n-1} p_{n-1} + \mu_{n+1} p_{n+1} = (\lambda_n + \mu_n) p_n$

Equilibrium State Equations

Birth-Death Process

- 由平衡方程式可得

$$p_1 = \frac{\lambda_0}{\mu_1} p_0$$

$$p_1 = \frac{\lambda_0}{\mu_1} p_0 \qquad p_2 = \frac{\lambda_1}{\mu_2} p_1 + \frac{1}{\mu_2}(\mu_1 p_1 - \lambda_0 p_0)$$

$$= \frac{\lambda_1}{\mu_2} p_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} p_0$$

- 由數學歸納法（mathematical induction）可得

$$p_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n \mu_{n-1}\cdots\mu_1} p_0$$

$$= p_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \qquad n = 1, 2, \cdots \qquad (3)$$

Birth-Death Process

- 因為

$$\sum_{n=0}^{\infty} p_n = p_0 + p_0 \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} = 1$$

- 所以

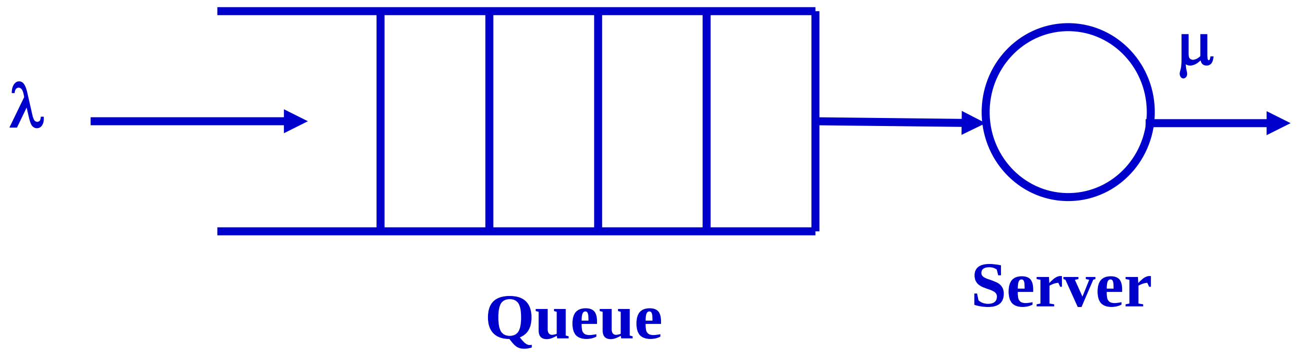
$$p_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}} \qquad (4)$$

- 以下所討論的等候模式，若屬生死過程，即可用式(3)及式(4)

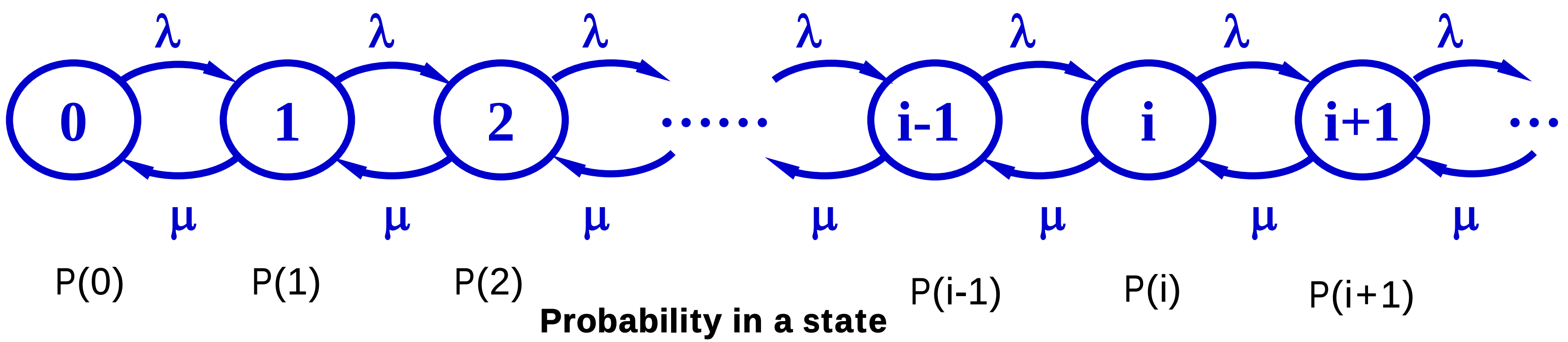
M/M/1/∞ or M/M/1 Queuing System

- **When a customer arrives in this system it will be served if the server is free, otherwise the customer is queued**
- **In this system customers arrive according to a *Poisson distribution* and compete for the service in a FIFO (first in first out) manner**
- **Service times are independent identically distributed (IID) random variables, the common distribution being exponential**

Queuing Model and State Transition Diagram



The M/M/1/∞ queuing model



The state transition diagram of the M/M/1/∞ queuing system

Equilibrium State Equations

- **If mean arrival rate is λ and mean service rate is μ, i = 0, 1, 2 be the number of customers in the system and *P(i)* be the state probability of the system having *i* customers**
- **From the state transition diagram, the equilibrium state equations are given by**

$$\lambda P(0) = \mu P(1), \qquad i = 0,$$

$$(\lambda + \mu) P(i) = \lambda P(i-1) + \mu P(i+1), \qquad i \geq 1$$

$$P(i) = \left(\tfrac{\lambda}{\mu}\right)^i P(0), \qquad i \geq 1$$

Equilibrium State Equations

$$
\begin{cases}
P(1) = \dfrac{\lambda}{\mu} P(0), \\
P(2) = \dfrac{\lambda}{\mu} P(1) = (\dfrac{\lambda}{\mu})^2 P(0) = \rho^2 P(0), \\
\cdots \\
P(i) = \dfrac{\lambda}{\mu} P(i-1) = (\dfrac{\lambda}{\mu})^t P(0) = \rho^i P(0),
\end{cases}
\qquad \rho = \frac{\lambda}{\mu}
$$

Traffic Intensity

- **We know that the *P(0)* is the probability of server being free. Since *P(0) > 0*, the necessary condition for a system being in steady state is,**

$$\rho = \frac{\lambda}{\mu} < 1$$

This means that the arrival rate cannot be more than the service rate, otherwise an infinite queue will form and jobs will experience infinite service time

M/M/1 Queuing System

P_n為系統中顧客數量 n 之機率，又機率總和等於 1，因此

$$\sum_{n=0}^{\infty} P_n = 1 \Rightarrow P_0 + P_1 + P_2 + \ldots + P_n + \ldots = 1$$

$$P_0 + (\lambda/\mu) P_0 + (\lambda/\mu)^2 P_0 + \ldots + (\lambda/\mu)^n P_0 + \ldots = 1$$

$$P_0 [\, 1 + (\lambda/\mu) + (\lambda/\mu)^2 + \ldots + (\lambda/\mu)^n + \ldots \,] = 1$$

Geometric Series (等比級數)

$$P_0 \left[\frac{1}{1-(\lambda/\mu)} \right] = 1$$

$P_0 = 1 - (\lambda/\mu) = 1 - \rho$, $\rho = \lambda/\mu$

$P_n = (\lambda/\mu)^n P_0 = \rho^n (1 - \rho)$, $n \geqq 0$

P_0為系統中沒有顧客之機率，即服務設施閒置之機率。

Queuing System Metrics

- $\rho = 1 - P(0)$, is **the probability of the server being busy**.
- **Therefore, we have**

$$P(n) = \rho^n (1 - \rho)$$

- **The average number of customers in the system is**

$$L_s = \frac{\lambda}{\mu - \lambda}$$

- **The average dwell time of customers is**

$$W_s = \frac{1}{\mu - \lambda}$$

$$L = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\,\rho^n (1-\rho)$$

$$= \Sigma\, n\,\rho^n - \Sigma\, n\,\rho^{n+1}$$

$$= (0 + \rho + 2\rho^2 + 3\rho^3 + \ldots)$$

$$- (0 + \rho^2 + 2\rho^3 + \ldots)$$

$$= \rho + \rho^2 + \rho^3 + \ldots$$

$$= \rho\,(1 + \rho + \rho^2 + \rho^3 + \ldots)$$ Geometric Series (等比級數)

$$= \rho \times \frac{1}{1-\rho}$$

$$= \frac{\lambda}{\mu - \lambda}$$ **(a)**

由(a)式 $L = \lambda W$ → **Little's Law**

$$\frac{\lambda}{\mu - \lambda} = \lambda W$$

$\therefore$

$$W = \frac{1}{\mu - \lambda} \quad \textbf{(c)}$$

由(c)式

$$W = W_q + 1/\mu$$

$$W_q = W - 1/\mu = \frac{\lambda}{\mu(\mu - \lambda)} \quad \textbf{(b)}$$

由(b)式

$$L_q = \lambda W_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

Queuing System Metrics

- **The average queuing length is**

$$Lq = \sum_{i=1}^{\infty} (i-1)P(i) = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

- **The average waiting time of customers is**

$$W_q = \frac{L_q}{\lambda} = \frac{\rho^2}{\lambda(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

Example

- 問題
  - 某郵局有一個專門辦理郵寄業務的窗口，中午12:00至下午1:00，到郵局辦理郵寄業務的顧客呈指數分配，平均每小時30人，每位顧客的服務時間亦呈指數分配，平均為1.5分鐘，求有五位顧客以上的機率為何?
- 解答
  - 此窗口的等候模式為$M/M/1$模式，且

$$\lambda = 30/\text{hr}$$

$$\frac{1}{\mu} = 1.5\,\text{min} = \frac{1}{40}\,\text{hr} \Rightarrow \mu = 40/\text{hr}$$

Example

- 因此

$$\rho = \frac{\lambda}{\mu} = \frac{30}{40} = 0.75$$

- 代入$M/M/1$模式的相關公式可得

$$L = \frac{\lambda}{\mu - \lambda} = \frac{30}{40 - 30} = 3\text{位顧客}$$

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{30^2}{40(40 - 30)} = 2.25\text{位顧客}$$

$$W = \frac{1}{\mu - \lambda} = \frac{1}{40 - 30} = 0.1\text{ hr} = 6\text{ min}$$

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{30}{40(40 - 30)} = 0.075\text{ hr} = 4.5\text{ min}$$

Example

- 代入 p_n 公式可得

$$p_n = \rho^n (1-\rho) = (0.75)^n (1-0.75)$$

- 因此

$$p_0 = 0.250,\ p_1 = 0.188,\ p_2 = 0.141,\ p_3 = 0.105,\ p_4 = 0.079$$

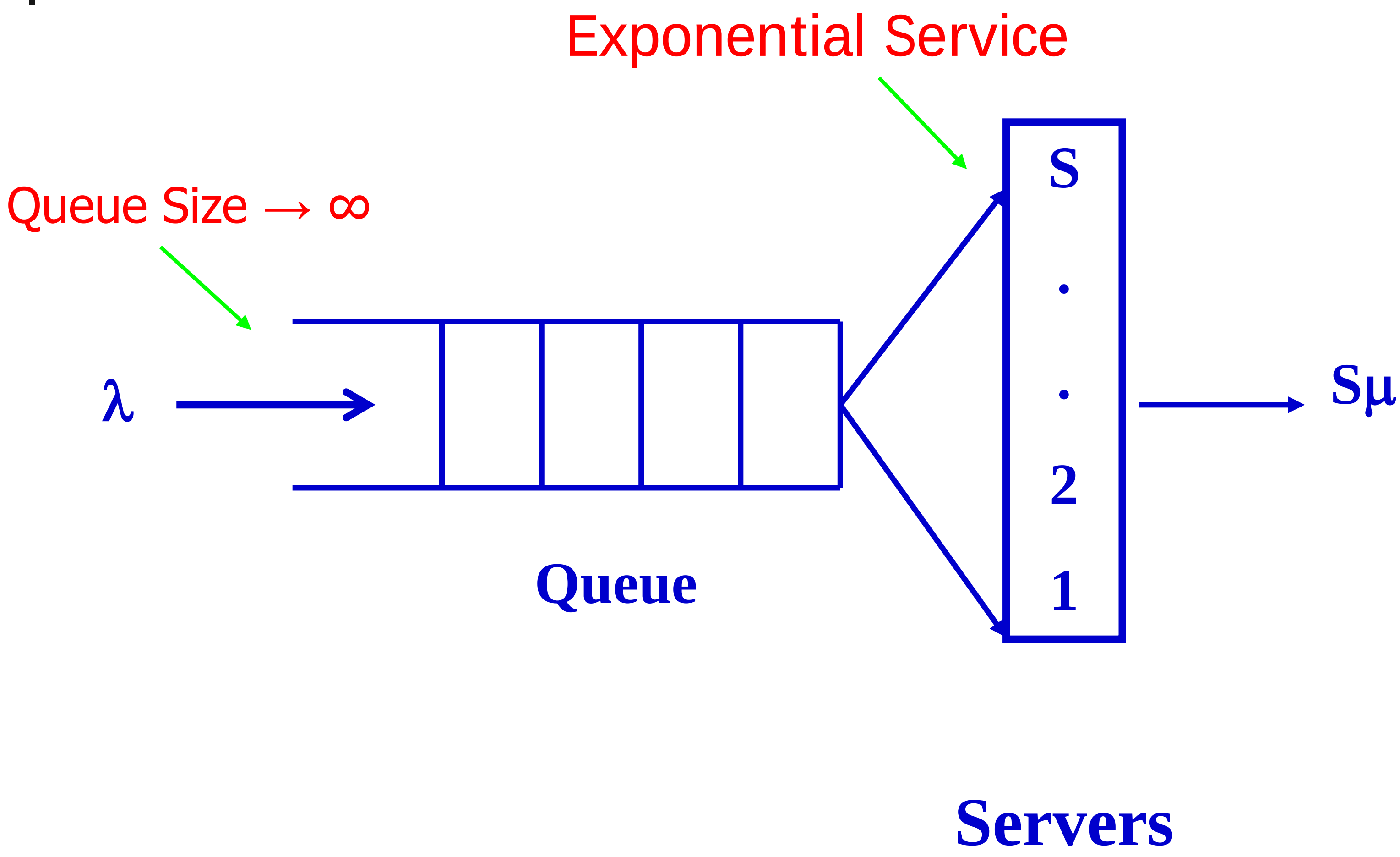
- 五位以上顧客的機率為

$$1-\sum_{n=0}^{4} p_n = 1-0.763 = 0.237$$

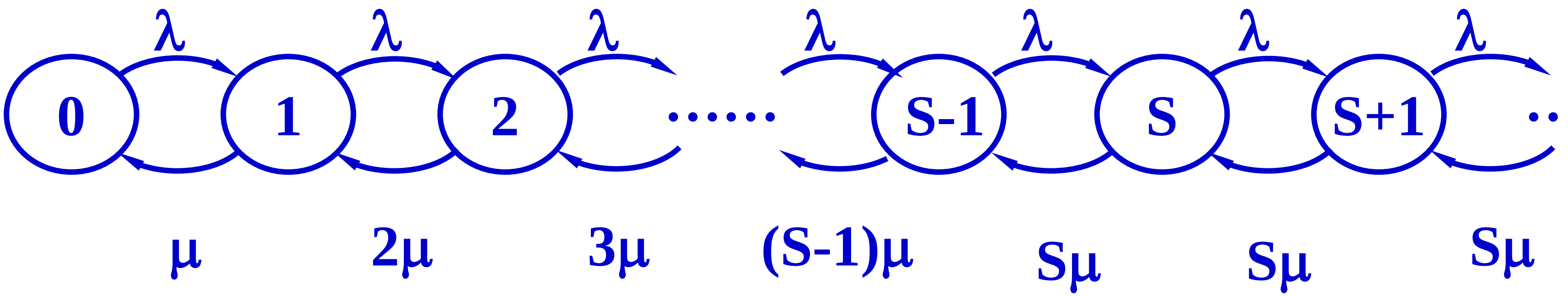
M/M/S/∞

- Customers arrive according to a Poisson process with rate λ.
- The service times of customers are exponentially distributed with parameter μ.
- There are ***s*** servers, serving customers in order of arrival.
- Stability condition: $\lambda < s * \mu$ or alternatively written, $\rho = \lambda / (s * \mu) < 1$.

M/M/S/∞ Queuing Model



State Transition Diagram



M/M/S/∞ Queuing Model

$$p_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} p_0$$

- 此模式的 λ_n 及 μ_n 如下：

$$= p_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \qquad n = 1, 2, \cdots$$

$$\lambda_n = \lambda \qquad n = 0, 1, \ldots$$

$$\mu_n = \begin{cases} n\mu & n = 1, 2, \ldots, s-1 \\ s\mu & n = s, s+1, \ldots \end{cases}$$

- 欲求得 p_n，我們先計算式(3)中的乘積部分如下：

$$\prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} = \begin{cases} \dfrac{\lambda^n}{(1\mu)(2\mu)\cdots(n\mu)} = \dfrac{(\lambda/\mu)^n}{n!} & n = 1, 2, \ldots, s-1 \\ \dfrac{\lambda^n}{(1\mu)(2\mu)\cdots(s\mu)(s\mu)^{n-s}} = \dfrac{(\lambda/\mu)^n}{s!s^{n-s}} & n = s, s+1, \ldots \end{cases}$$

M/M/S/∞ Queuing Model

- Thus，

$$p_n = \begin{cases} \dfrac{(\lambda/\mu)^n}{n!} p_0 & n = 1, 2, \ldots, s-1 \\ \dfrac{(\lambda/\mu)^n}{s!\,s^{n-s}} p_0 & n = s, s+1, \ldots \end{cases}$$

$$p_0 = \left[1 + \sum_{n=1}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!} \sum_{n=s}^{\infty} \left(\frac{\lambda}{s\mu} \right)^{n-s} \right]^{-1}$$

$$= \left[\sum_{n=0}^{s-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!} \frac{1}{1 - \lambda/(s\mu)} \right]^{-1}$$

Geometric Series (等比級數)

- 以上 p_0 的求導過程中，必須 $\rho = \lambda / s\mu < 1$

M/M/S/∞

- 至於四個績效基準，可計算 L_q 如下：

$$L_q = \sum_{n=s}^{\infty}(n-s)p_n$$

$$= \sum_{i=0}^{\infty} i p_{s+i} = \sum_{i=0}^{\infty} i \frac{(\lambda/\mu)^s}{s!}\rho^i p_0$$

$$= p_0 \frac{(\lambda/\mu)^s}{s!}\rho \sum_{i=0}^{\infty} i\rho^{i-1} = p_0 \frac{(\lambda/\mu)^s}{s!}\rho \sum_{i=0}^{\infty} \frac{d}{d\rho}\rho^i$$

$$= p_0 \frac{(\lambda/\mu)^s}{s!}\rho \frac{d}{d\rho}\sum_{i=0}^{\infty}\rho^i = p_0 \frac{(\lambda/\mu)^s}{s!}\rho \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right)$$

$$= \frac{(\lambda/\mu)^s \rho}{s!(1-\rho)^2} p_0$$

- 其餘的績效基準可由 Little 公式及基本關係求得

Queuing System Metrics

- The average number of customers in the system is

$$L = \lambda(W_q + \frac{1}{\mu}) = L_q + \frac{\lambda}{\mu}$$

$$\frac{(\lambda / \mu)^s \rho}{s!(1-\rho)^2} p_0$$

- The average waiting time of customers is

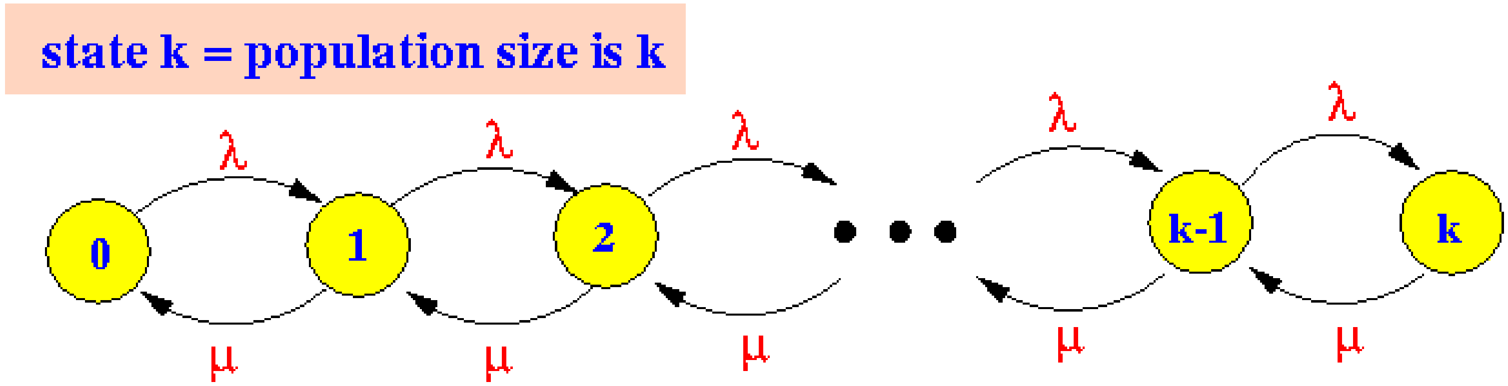
$$W_q = \frac{L_q}{\lambda}$$

- The average dwell time of a customer in the system is given by

$$W = W_q + \frac{1}{\mu}$$

M/M/1/K

- Arrival process is Poisson Process with rate λ
- Number of services is Poisson Process with rate μ
- Single Server
- Queue size is finite = K
- Queue discipline : FCFS



M/M/1/K

- 到達率與服務率為

$$\lambda_n = \begin{cases} \lambda & n = 0, 1, \ldots, K-1 \\ 0 & n = K, K+1, \ldots \end{cases}$$

$$\mu_n = \mu \qquad n = 1, 2, \cdots$$

$$p_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} p_0$$

$$= p_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \qquad n = 1, 2, \cdots \qquad (3)$$

$$p_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}} \qquad (4)$$

- 將 λ_n 與 μ_n 代入式(4)及式(3)，分別可得

$$p_0 = \frac{1}{\rho^0 + \sum_{n=1}^{K} \rho^n} = \frac{1}{\sum_{n=0}^{K} \rho^n}$$

$$= \begin{cases} \dfrac{1-\rho}{1-\rho^{K+1}} & \rho \neq 1 \\ \dfrac{1}{K+1} & \rho = 1 \end{cases}$$

$$P_n = \rho^n \frac{1-\rho}{1-\rho^{k+1}}$$

M/M/1/K

$$p_n = \rho^n p_0$$

$$= \begin{cases} \dfrac{1-\rho}{1-\rho^{K+1}}\rho^n & \rho \neq 1 \\[2ex] \dfrac{1}{K+1} & \rho = 1 \end{cases} \qquad \text{for } n = 0, 1, \ldots, K$$

- 在以上 p_0 推導中，用到有限總和（finite sum）公式：

$$\sum_{n=0}^{K} x^n = \begin{cases} \dfrac{1-x^{K+1}}{1-x} & x \neq 1 \\[2ex] K+1 & x = 1 \end{cases}$$

For $r \neq 1$, the sum of the first n terms of a geometric series is

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\,\frac{1-}{1-}$$

- 此模式的穩定狀態並不需要 $\rho = \lambda / \mu < 1$

M/M/1/K

$$L = \sum_{k=1}^{K} k\rho^k P_0 = \rho P_0 \sum_{k=1}^{K} k\rho^{k-1}$$

$$= \rho P_0 \left(\sum_{k=1}^{K} \rho^k\right)' = \rho P_0 \left(\rho \frac{1-\rho^K}{1-\rho}\right)' = \rho P_0 \left(\frac{\rho - \rho^{K+1}}{1-\rho}\right)'$$

$$= \left(\left(1-(K+1)\rho^K\right)(1-\rho) + \rho - \rho^{K+1}\right) \cdot \frac{\rho P_0}{(1-\rho)^2}$$

$$= \frac{\rho P_0 \left(1-(K+1)\rho^K - \rho + (K+1)\rho^{K+1} + \rho - \rho^{K+1}\right)}{(1-\rho)^2}$$

$$= \frac{\rho P_0 \left(1-(K+1)\rho^K + K\rho^{K+1}\right)}{(1-\rho)^2}$$

$$= \frac{\rho \left(1-(K+1)\rho^K + K\rho^{K+1}\right)}{(1-\rho)(1-\rho^{K+1})}.$$

Quotient rule

(Quotient Rule) $\dfrac{d}{dx}\left(\dfrac{f}{g}\right)=\dfrac{\dfrac{df}{dx}\cdot g-f\cdot\dfrac{dg}{dx}}{g^2}$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{\dfrac{df(x)}{dx}\cdot g(x)-f(x)\cdot\dfrac{dg(x)}{dx}}{g^2(x)}$$

$$=\frac{f'(x)\cdot g(x)-f(x)\cdot g'(x)}{g^2(x)}$$

$$\frac{1}{(1-\rho)^2}\times\left((1-\rho)\times\left(1-(k+1)\rho^k\right)-\left(\rho-\rho^{k+1}\right)\times(-1)\right)$$

$$\frac{1}{(1-\rho)^2}\times\left((1-\rho)\times\frac{d}{d\rho}\left(\rho-\rho^{k+1}\right)-\left(\rho-\rho^{(k+1)}\right)\times\frac{d}{d\rho}(1-\rho)\right)$$

M/M/1/K

- 當 $\rho \neq 1$ 時：

$$L = \ldots$$

$$= \frac{\rho[K\rho^{K+1} - (K+1)\rho^{K} + 1]}{(1-\rho^{K+1})(1-\rho)}$$

- 若 $\rho = 1$，則

$$L = \sum_{n=0}^{K} n p_n = \frac{1}{K+1} \frac{K(K+1)}{2} = \frac{K}{2}$$

- L_q 可計算如下：

$$L_q = L - (1 - p_0)$$

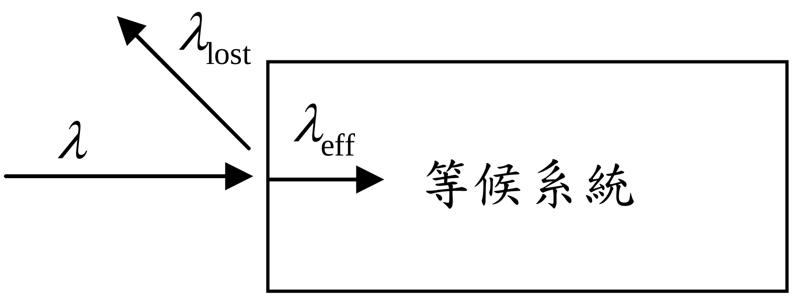
M/M/1/K

- 利用 Little 公式時，須使用有效到達率(effective arrival rate）λ_{eff} ：

$$\lambda_{\text{eff}} = \lambda(1 - p_K)$$

- 代入 Little 公式可得

$$W = \frac{L}{\lambda_{\text{eff}}} \qquad W_q = \frac{L_q}{\lambda_{\text{eff}}}$$



M/M/S/K

- 到達率與服務率為：

$$\lambda_n = \begin{cases} \lambda & n = 0, 1, \ldots, K-1 \\ 0 & n = K, K+1, \ldots \end{cases}$$

$$\mu_n = \begin{cases} n\mu & n = 1, 2, \ldots, s-1 \\ s\mu & n = s, s+1, \ldots, K \end{cases}$$

- 將 λ_n 與 μ_n 代入式(3)及式(4)，分別可得

$$p_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} p_0 = p_0 \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}} \qquad n = 1, 2, \cdots \qquad (3)$$

$$p_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_i}{\mu_{i+1}}} \qquad (4)$$

$$p_n = \begin{cases} \frac{(\lambda/\mu)^n}{n!} p_0 & n = 1, 2, \ldots, s-1 \\ \frac{(\lambda/\mu)^n}{s! s^{n-s}} p_0 & n = s, s+1, \ldots, K \\ 0 & n = K+1, \ldots \end{cases}$$

M/M/S/K

For $r \neq 1$, the sum of the first n terms of a geometric series is

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\frac{1-r^n}{1-r},$$

$$p_0 = \left[1 + \sum_{n=1}^{s-1}\frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!}\sum_{n=s}^{K}\left(\frac{\lambda}{s\mu}\right)^{n-s}\right]^{-1}$$

$$= \left[1 + \sum_{n=1}^{s-1}\frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!}\frac{1-(\lambda/s\mu)^{K-s+1}}{1-(\lambda/s\mu)}\right]^{-1}$$

- 得到所有 p_n 後，即可求得 L_q（當 $\rho = \lambda/s\mu \neq 1$ 時）如下：

$$L_q = \sum_{n=s}^{K}(n-s)p_n = \ldots$$

$$= \frac{(\lambda/\mu)^s \rho}{s!(1-\rho)^2} p_0 [(K-s)(\rho-1)\rho^{K-s} - \rho^{K-s} + 1]$$

M/M/S/∞

M/M/S/K

- 此模式之 L 與 L_q 的關係如下：

$$L = L_q + \sum_{n=0}^{s-1} n p_n + s\left(1 - \sum_{n=0}^{s-1} p_n\right)$$

- 欲利用 Little 公式，須使用有效到達率 λ_{eff} ：

$$\lambda_{\text{eff}} = \lambda(1 - p_K)$$

- 因此，

$$W = \frac{L}{\lambda_{\text{eff}}} \qquad W_q = \frac{L_q}{\lambda_{\text{eff}}}$$

Example

- **概述**
- 保養廠設置2個升降工作台（各1位維修員），並可停放3輛
- 若汽車無法進入保養廠停放，將會離開
- 到達率呈指數分配，平均每小時2輛
- 維修時間呈指數分配，平均需要40分鐘
- 每位顧客平均消費金額$1350

- **問題**

a. 廠內有n位顧客的機率
b. 有效到達率
c. 每天營業的10小時期間，因顧客無法進入而損失的營業額
d. 兩工作台的期望車輛數
e. 每位維修員每天空閒時間的百分比
f. 等候維修的期望車輛數
g. 每位顧客在保養廠內的期望時間

Solution

(a)
- 此為 $M/M/s/K$ 模式，其 $s=2, K=5$，且

$$\lambda_n = 2/\text{hr} \qquad n = 0,1,\ldots,4$$

$$\frac{1}{\mu} = 40\,\text{min} = \frac{2}{3}\text{hr} \Rightarrow \mu = 1.5/\text{hr}$$

$$\mu_n = \begin{cases} 1.5n & n = 1,2 \\ 3 & n = 3,4,5 \end{cases}$$

- 代入 p_n 公式可得

$$p_n = \begin{cases} \dfrac{(2/1.5)^n}{n!} p_0 & n = 1,2 \\ \dfrac{(2/1.5)^n}{2!2^{n-s}} p_0 & n = 3,4,5 \end{cases}$$

Solution

(a)

- 因此，

$$p_1 = \frac{4}{3}p_0,\ p_2 = \frac{1}{2}\left(\frac{4}{3}\right)^2 p_0,\ p_3 = \frac{1}{4}\left(\frac{4}{3}\right)^3 p_0$$

$$p_4 = \frac{1}{8}\left(\frac{4}{3}\right)^4 p_0,\ p_5 = \frac{1}{16}\left(\frac{4}{3}\right)^5 p_0$$

- 因機率總和等於1，所以

$$p_0\left(1+\frac{4}{3}+\frac{1}{2}\left(\frac{4}{3}\right)^2+\frac{1}{4}\left(\frac{4}{3}\right)^3+\frac{1}{8}\left(\frac{4}{3}\right)^4+\frac{1}{16}\left(\frac{4}{3}\right)^5\right)=1$$

- 計算可得 $p_0 = 0.2236$ ，因此

$$p_1 = 0.2981,\ p_2 = 0.1988,\ p_3 = 0.1325,\ p_4 = 0.0883,\ p_5 = 0.058$$

Solution

(b)

- 有效到達率可計算如下：

$$\lambda_{\text{eff}} = \lambda(1-p_5) = 2(1-0.0589) = 1.8822 \text{ cars / hr}$$

(c)

- 每小時無法進入的車輛數為

$$\lambda p_5 = 2(0.0589) = 0.1178 \quad \text{cars}$$

- 因此每天 10 小時所損失的營業額為

$$\$1350 \times 10 \times \lambda p_5 = \$1590.3$$

Solution

(d) 兩工作台的期望車輛數為

$$1p_1 + 2(p_2 + p_3 + p_4 + p_5) = 1.2551 \text{ cars}$$

(e) 每位維修員每天空閒時間的百分比可計算如下：

$$(2p_0 + 1p_1)/2 = 37.27\%$$

(f) 等候維修的期望車輛數為

$$L_q = \sum_{n=3}^{5}(n-2)p_n = 1p_3 + 2p_4 + 3p_5 = 0.4858 \text{ cars}$$

Solution

(g) 每位顧客在保養廠內的期望時間為

$$W_q = \frac{L_q}{\lambda_{\text{eff}}} = \frac{0.4858}{1.8822} = 0.2581 \text{ hrs} = 15.49 \text{ min}$$

$$W = W_q + \frac{1}{\mu} = 55.49 \text{ min}$$

Example

- 某工廠買了許多同一型式之機器，現在需要確定一名工人應看管幾台機器，機器在正常運轉時是不需要看管的。已知每台機器的正常運轉時間服從平均數為120分鐘的指數分配，工人看管一台機器的時間服從平均數為12分鐘的指數分配。每名工人只能看管自己的機器，工廠要求每台機器的正常運轉時間不得少於87.5％。問在此條件之下每名工人最多能看管幾台機器？

Solution

Sol:這是M/M/1,有限來源排隊系統。每名工人看管最多台數爲 k

$\lambda = \frac{1}{2}, \mu = 5$

$\therefore \rho = \frac{1}{10}$

因此，

$$L_s = k - \frac{\mu}{\lambda}(1 - P_0) = k - \rho(1 - P_0)$$

$$P_0 = \left[\sum_{n=0}^{k} \frac{k!}{(k-n)!} \rho^n \right]^{-1}$$

根據要求，停止運轉的機器數 $L_s \le 0.125k$。當m=1,2,3,4,5,時如以下之結果：

k	P_0	L_s	$L_s \le 0.125k$?
1	10/11	0.091	是(0.125)
2	50/61	0.197	是(0.25)
3	500/683	0.321	是(0.375)
4	1250/1933	0.467	是(0.5)
5	2500/4433	0.640	否(0.625)

可見一名工人最多只能看管四台機器

Example

- 某戲院有三個售票窗口，每個售票窗口對於每位客人之平均出售時間為5分鐘，依指數分佈；到達該戲院每小時平均6人，依Poisson分佈。但等候列的長度為30個人，試求下列問題？

  （１） 系列中有n個客人的機率

  （２） 系列之呼損率（系列中已滿，再到達之客人不能進入系列中而馬上離開之機率）

Solution

Sol:此題爲M/M/3/30，S=3，K=30，$\lambda=6$（人/小時）

$$\mu=\tfrac{60}{5}(\text{人/分})=12(\text{人/小時})$$

$$\rho=\tfrac{\lambda}{3\mu}=\tfrac{1}{6}<1,\text{先求}P_0$$

$$\sum_{n=0}^{3}\frac{1}{n!}(1/2)^n+\frac{3^3}{3!}\sum_{n=4}^{30}(1/6)^n$$

$$=1+\left(\tfrac{1}{2}\right)+\tfrac{1}{2}\left(\tfrac{1}{2}\right)^2+\tfrac{1}{6}\left(\tfrac{1}{2}\right)^3+\tfrac{9}{2}\times\frac{\left(\frac{1}{6}\right)^4\left[1-\left(\frac{1}{6}\right)^{27}\right]}{1-\frac{1}{6}}$$

$$=1.65$$

$$P_0=\tfrac{1}{1.65}=0.61$$

（1） $$P_n=\begin{cases}\dfrac{1}{n!}\left(\dfrac{1}{6}\right)^n\times 0.61, & 0\le n<3\\[2ex] \dfrac{1}{3!3^{n-3}}\left(\dfrac{1}{6}\right)^n\times 0.61, & 3\le n<30\end{cases}$$

（2） $$P_N=\frac{\rho^N}{s!s^{N-s}}P_0=\frac{1}{3!3^{27}}\left(\frac{1}{6}\right)^{30}\times 0.61=0$$

HW

- 有一超級市場在顧客的期望下，該店經理相信顧客對需排隊等候8分鐘並且共花10分鐘在等候系統上（不含真正購物時間）是難以接受的。因此經理嘗試下列兩方案中的一個來縮短顧客等待時間：（1）增聘一位員工打包貨品；（2）增加一個結帳櫃臺。