Chapter 4 Poisson Processes - 2

Non-homogeneous Poisson Processes

- The counting process $N = \{\tilde{n}(t), t \geq 0\}$ is said to be a *non-stationary* or *non-homogeneous* Poisson Process with time-varying intensity function $\lambda(t), t \geq 0$, if: 與時間相關

  1. $\tilde{n}(0) = 0$
  2. N has independent increments
  3. $P[\tilde{n}(t+h) - \tilde{n}(t) \geq 2] = o(h)$
  4. $P[\tilde{n}(t+h) - \tilde{n}(t) = 1] = \lambda(t) \cdot h + o(h)$

$$P(X(h)=1) = e^{-\lambda h} (\lambda h)$$
$$= \lambda h(1 - \lambda h + o(h))$$
$$= \lambda h + o(h)$$

- Define "integrated intensity function" $m(t) = \int_0^t \lambda(t')dt'$.

Theorem.

$$\boxed{P[\tilde{n}(t+s) - \tilde{n}(t) = n] = \frac{e^{-[m(t+s)-m(t)]}[m(t+s) - m(t)]^n}{n!}}$$

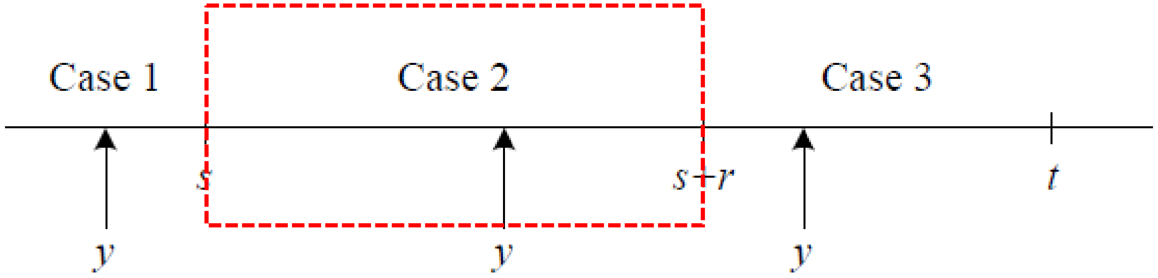
Proof. < Homework >.

Non-homogeneous Poisson Processes

- **Example.** The "output process" of the *M/G/∞* queue is a non-homogeneous Poisson process having intensity function $\lambda(t) = \lambda G(t)$, where G is the service distribution.
- **Hint.** Let $D(s, s + r)$ denote the number of service completions in the interval $(s, s + r]$ in $(0, t]$. If we can show that
  - $D(s, s + r)$ follows a Poisson distribution with mean $\lambda \int_s^{s+r} G(y)dy$ and
  - the numbers of service completions in disjoint intervals are independent,
- Then we are finished by definition of a non-homogeneous Poisson process

Non-homogeneous Poisson Processes

- **Answer.**
  - An arrival at time *y* is called a type-1 arrival if its service completion occurs in (*s*, *s* + *r*].
  - Consider three cases to find the probability *P*(*y*) that an arrival at time *y* is a type-1 arrival:



– Case 1: $y \leq s$.

$$P(y) = P\{s - y < \tilde{S} < s + r - y\} = G(s + r - y) - G(s - y)$$

– Case 2: $s < y \leq s + r$.

$$P(y) = P\{\tilde{S} < s + r - y\} = G(s + r - y)$$

Non-homogeneous Poisson Processes

– Case 3: $s + r < y \le t$.

$$P(y) = 0$$

- Based on the decomposition property of a Poisson process, we conclude that *D*(*s*, *s* + *r*) follows a Poisson distribution with mean λpt, where $p = (1/t)\int_0^t P(y)dy$

$$\begin{aligned}
\int_0^t P(y)dy &= \int_0^s [G(s+r-y) - G(s-y)]dy + \int_s^{s+r} G(s+r-y)dy \\
&\quad + \int_{s+r}^t (0)dy \\
&= \int_0^{s+r} G(s+r-y)dy - \int_0^s G(s-y)dy \\
&= \int_0^{s+r} G(z)dz - \int_0^s G(z)dz = \int_s^{s+r} G(z)dz
\end{aligned}$$

Non-homogeneous Poisson Processes

- Because of
  - the independent increment assumption of the Poisson arrival process, and
  - the fact that there are always servers available for arrivals,

$\Rightarrow$ the departure process has independent increments

Compound Poisson Processes

- A stochastic process $\{\tilde{x}(t), t \geq 0\}$ is said to be a *compound Poisson process* if
  - it can be represented as
    $$\boxed{\tilde{x}(t) = \sum_{i=1}^{\tilde{n}(t)} \tilde{y}_i, \quad t \geq 0}$$
  - $\{\tilde{n}(t), t \geq 0\}$ is a Poisson process
  - $\{\tilde{y}_i, i \geq 1\}$ is a family of independent and identically distributed random variables which are also independent of $\{\tilde{n}(t), t \geq 0\}$
- The random variable $\tilde{x}(t)$ is said to be a compound Poisson random variable.
- $E[\tilde{x}(t)] = \lambda t E[\tilde{y}_i]$ and $Var[\tilde{x}(t)] = \lambda t E[\tilde{y}_i{}^2]$.

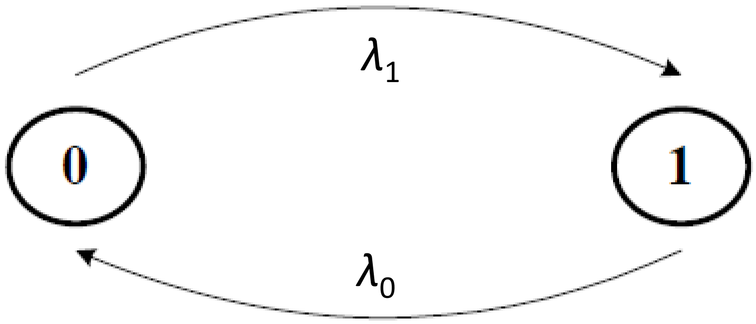
Compound Poisson Processes

- **Example** (Batch Arrival Process). Consider a parallel-processing system where each job arrival consists of a possibly random number of tasks. Then we can model the arrival process as a *compound Poisson process*, which is also called a *batch arrival process*.
- Let $\tilde{y}_i$ be a random variable that denotes *the number of tasks comprising a job*. We derive the probability generating function $P_{\tilde{x}(t)}(z)$ as follows:

$$
\begin{aligned}
P_{\tilde{x}(t)}(z) &= E\left[z^{\tilde{x}(t)}\right] = E\left[E\left[z^{\tilde{x}(t)} | \tilde{n}(t)\right]\right] = E\left[E\left[z^{\tilde{y}_1+\cdots+\tilde{y}_{\tilde{n}(t)}} | \tilde{n}(t)\right]\right] \\
&= E\left[E\left[z^{\tilde{y}_1+\cdots+\tilde{y}_{\tilde{n}(t)}}\right]\right] \quad \text{(by independence of } \tilde{n}(t) \text{ and } \{\tilde{y}_i\}) \\
&= E\left[E\left[z^{\tilde{y}_1}\right]\cdots E\left[z^{\tilde{y}_{\tilde{n}(t)}}\right]\right] \quad \text{(by independence of } \tilde{y}_1, \cdots, \tilde{y}_{\tilde{n}(t)}) \\
&= E\left[(P_{\tilde{y}}(z))^{\tilde{n}(t)}\right] = P_{\tilde{n}(t)}\left(P_{\tilde{y}}(z)\right)
\end{aligned}
$$

Modulated Poisson Processes

- Assume that there are two states, 0 and 1, for a “modulating process.”



- When the state of the modulating process equals 0 then the arrive rate of customers is given by λ_0, and when it equals 1 then the arrival rate is λ_1.
- The residence time in a particular modulating state is exponentially distributed with parameter μ and, after expiration of this time, the modulating process changes state.
- The initial state of the modulating process is randomly selected and is equally likely to be state 0 or 1.

Modulated Poisson Processes

待在state 0 的時間

- For a given period of time (0, *t*), let Υ be a random variable that indicates the total amount of time that the modulating process has been in state 0. Let $\tilde{x}(t)$be the number of arrivals in (0, *t*).
- Then, given Υ, the value of $\tilde{x}(t)$is distributed as a non-homogeneous Poisson process and thus

$$P[\tilde{x}(t) = n | \Upsilon = \tau] = \frac{(\lambda_0 \tau + \lambda_1 (t - \tau))^n e^{-(\lambda_0 \tau + \lambda_1 (t - \tau))}}{n!}$$

- As $\mu \to 0$, the probability that the modulating process makes no transitions within t seconds converges to 1, and we expect for this case that

$$P[\tilde{x}(t) = n] = \frac{1}{2} \left\{ \frac{(\lambda_0 t)^n e^{-\lambda_0 t}}{n!} + \frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!} \right\}$$

Modulated Poisson Processes

- As $\mu \to \infty$, then the modulating process makes an infinite number of transitions within t seconds, and we expect for this case that

$$P[\tilde{x}(t) = n] = \frac{(\beta t)^n e^{-\beta t}}{n!}, \quad \text{where } \beta = \frac{\lambda_0 + \lambda_1}{2}$$

- **Example** (Modeling Voice).
- A basic feature of speech is that it comprises an alternation of silent periods and non-silent periods.
- The arrival rate of packets during a talk spurt period is Poisson with rate λ_1 and silent periods produce a Poisson rate with $\lambda_0 \approx 0$.
- The duration of time for talk and silent periods are exponentially distributed with parameters μ_1 and μ_0, respectively.

$\Rightarrow$ The model of the arrival stream of packets is given by a modulated Poisson process.

Poisson Arrivals See Time Averages (PASTA)

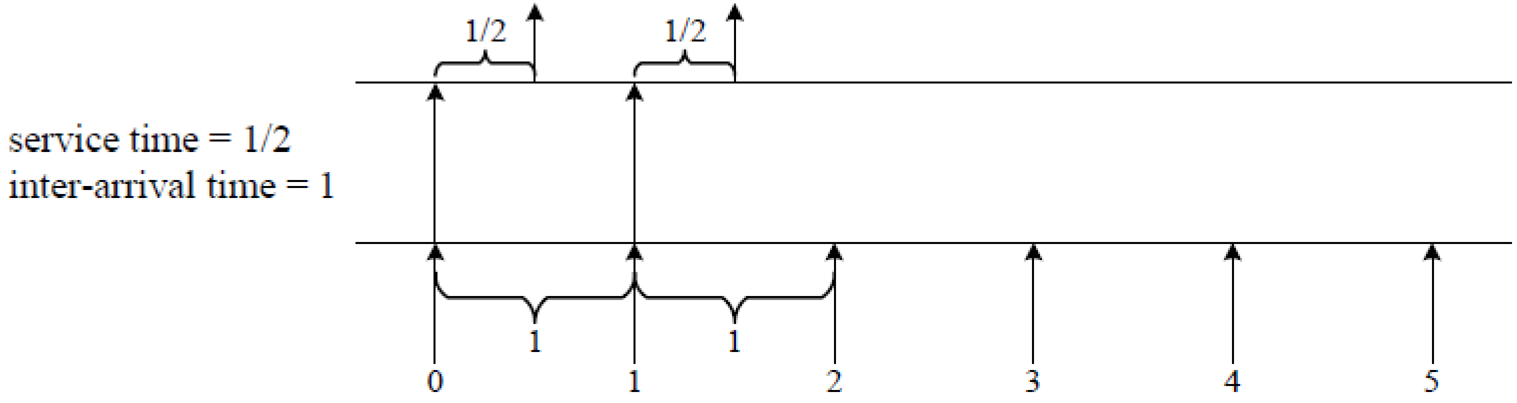
- PASTA says: as $t \to \infty$

  Fraction of arrivals who see the system in a given state upon arrival (arrival average)

  = Fraction of time for the system is in a given state (time average)

  = The system is in the given state at any random time after being steady

Counter-example (textbook [Kao]: Example 2.7.1)



Poisson Arrivals See Time Averages (PASTA)

- Arrival average that an arrival will see an idle system = 1
- Time average of system being idle = ½

Mathematically,

- Let $X = \{\tilde{x}(t), t \geq 0\}$ be a stochastic process with state space S, and $B \subset S$
- Define an indicator random variable $$\tilde{u}(t) = \begin{cases} 1, & \text{if } \tilde{x}(t) \in B \\ 0, & \text{otherwise} \end{cases}$$
- Let $N = \{\tilde{n}(t), t \geq 0\}$ be a Poisson process with rate λ denoting the arrival process

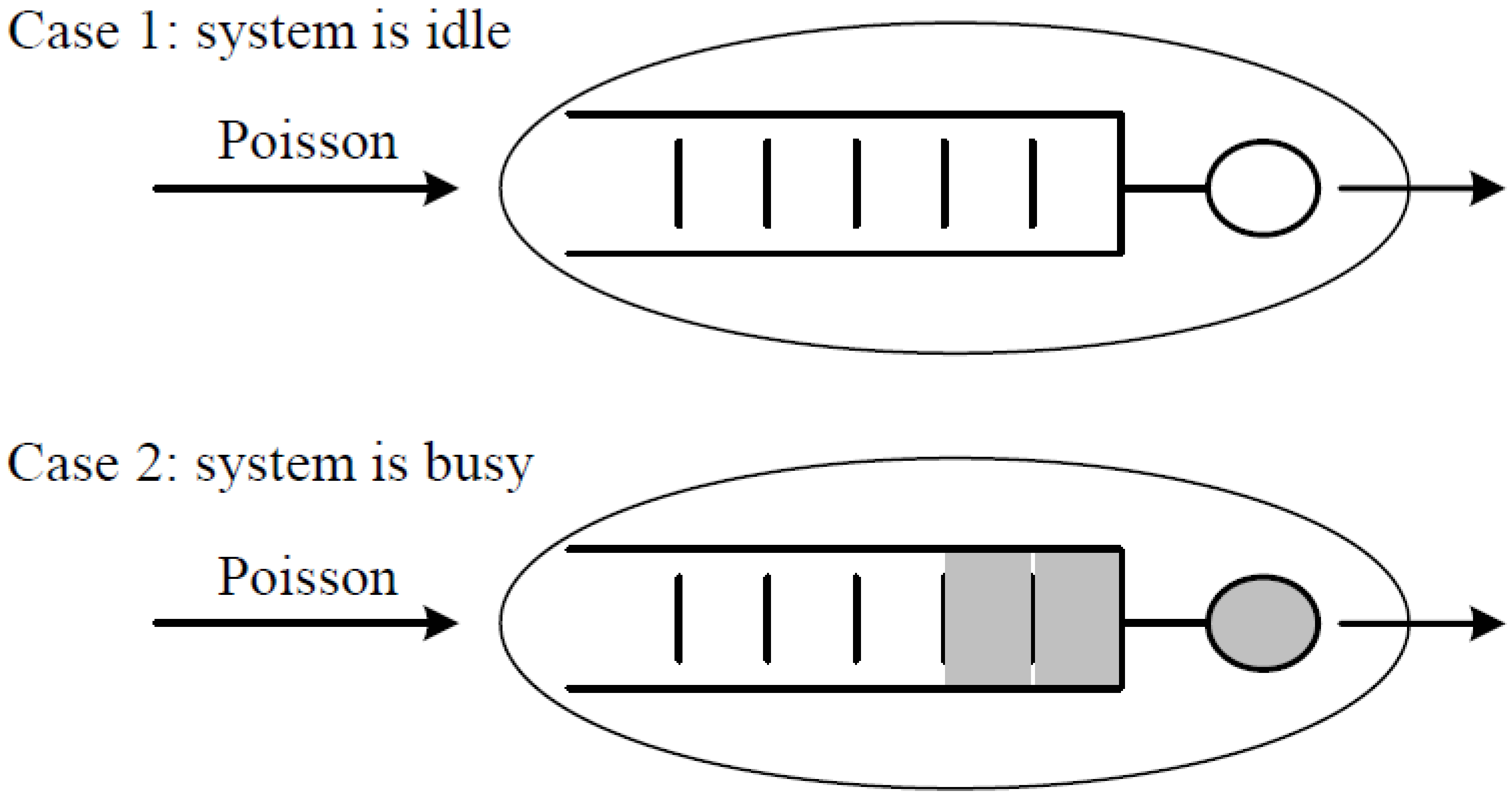
Poisson Arrivals See Time Averages (PASTA)

$$\lim_{t\to\infty} \frac{\int_0^t \tilde{u}(s)d\tilde{n}(s)}{\tilde{n}(t)} = \lim_{t\to\infty} \frac{\int_0^t \tilde{u}(s)ds}{t}$$

$$(arrival\ average) \qquad\qquad (time\ average)$$

- Condition – For PASTA to hold, we need the *lack of anticipation assumption* (LAA): for each $t \geq 0$,
  - the arrival process $\{\tilde{n}(t+u) - \tilde{n}(t),\ u \geq 0\}$ is independent of $\{\tilde{x}(s),\ 0 \leq s \leq t\}$ and $\{\tilde{n}(s),\ 0 \leq s \leq t\}$.
- Application:
  - To find the waiting time distribution of any arriving customer
  - Given: P[system is idle] = $1 - \rho$; P[system is busy] = ρ

Poisson Arrivals See Time Averages (PASTA)



$$\begin{aligned}\Rightarrow P(\tilde{w} \leq t) &= P(\tilde{w} \leq t|\text{idle}) \cdot P(\text{idle upon arrival}) \\ &+ P(\tilde{w} \leq t|\text{busy}) \cdot P(\text{busy upon arrival})\end{aligned}$$

Memoryless Property of the Exponential Distribution

- A random variable $\tilde{x}$ is said to be without memory, or ***memoryless***, if

$$P[\tilde{x} > s + t | \tilde{x} > t] = P[\tilde{x} > s] \quad \text{for all } s, t \geq 0 \tag{3}$$

- The condition in Equation (3) is equivalent to

$$\frac{P[\tilde{x} > s + t, \tilde{x} > t]}{P[\tilde{x} > t]} = P[\tilde{x} > s]$$

or

$$P[\tilde{x} > s + t] = P[\tilde{x} > s]P[\tilde{x} > t] \tag{4}$$

- Since Equation (4) is satisfied when $\tilde{x}$ is exponentially distributed (for $e^{-\lambda(s+t)} = e^{-\lambda s}e^{-\lambda t}$), it follows that exponential random variable are memoryless.
- Not only is the exponential distribution "memoryless," but it is the unique continuous distribution possessing this property.

Exponential Distribution

- T_2 is the time between first and second arrivals, we define T_3 as the time between the second and third arrivals, T_4 as the time between the third and fourth arrivals and so on
- The random variables T_1, T_2, T_3… are called the <u>inter-arrival times of the Poisson process</u>
- T_1, T_2, T_3,… are *independent* of each other and each has the <u>same exponential distribution</u> with <u>mean arrival rate λ</u>

Memoryless and Merging Properties

- Memoryless property
  - A random variable X has the property that "*the future is independent of the past*" i.e., the fact that it hasn't happened yet, tells us nothing about how much longer it will take before it does happen
- Merging property
  - If we merge n Poisson processes with distributions for the inter arrival times

$$1- e^{-\lambda t} \text{ where } i = 1, 2, \ldots, n$$

into one single process, then the result is a Poisson process for which the inter arrival times have the distribution $1- e^{-\lambda t}$ with mean

$$\lambda = \lambda_1 + \lambda_2 +..+ \lambda_n$$

Memoryless

- **Proof：**

$$P\{T > t + t_0 \mid T > t_0\} = \frac{P\{T > t + t_0, T > t_0\}}{P\{T > t_0\}} = \frac{P\{T > t + t_0\}}{P\{T > t_0\}}$$

$$= \frac{e^{-\alpha(t+t_0)}}{e^{-\alpha t_0}} = e^{-\alpha t}$$

$$= P\{T > t\}$$

Comparison of Two Exponential Random Variables

- Suppose that $\tilde{x}_1$and $\tilde{x}_2$ are independent exponential random variables with respective means $1/\lambda_1$ and $1/\lambda_2$. What is $P[\tilde{x}_1 < \tilde{x}_2]$?

$$\begin{aligned} P[\tilde{x}_1 < \tilde{x}_2] &= \int_0^\infty P[\tilde{x}_1 < \tilde{x}_2 | \tilde{x}_1 = x] \lambda_1 e^{-\lambda_1 x} dx \\ &= \int_0^\infty P[x < \tilde{x}_2] \lambda_1 e^{-\lambda_1 x} dx \\ &= \int_0^\infty e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx \\ &= \int_0^\infty \lambda_1 e^{-(\lambda_1 + \lambda_2) x} dx \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

Minimum of Exponential Random Variables

- Suppose that $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n$ are independent exponential random variables, with $\tilde{x}_i$ having rate μ_i, $i = 1, \cdots, n$. It turns out that the smallest of the $\tilde{x}_i$ is exponential with a rate equal to the sum of the μ_i.

$$\begin{aligned}
P[\min(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n) > x] &= P[\tilde{x}_i > x \text{ for each } i = 1, \cdots, n] \\
&= \prod_{i=1}^{n} P[\tilde{x}_i > x] \quad \text{(by independence)} \\
&= \prod_{i=1}^{n} e^{-\mu_i x} \\
&= exp\left\{-\left(\sum_{i=1}^{n} \mu_i\right) x\right\}
\end{aligned}$$

How about $\max(\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_n)$? (exercise)

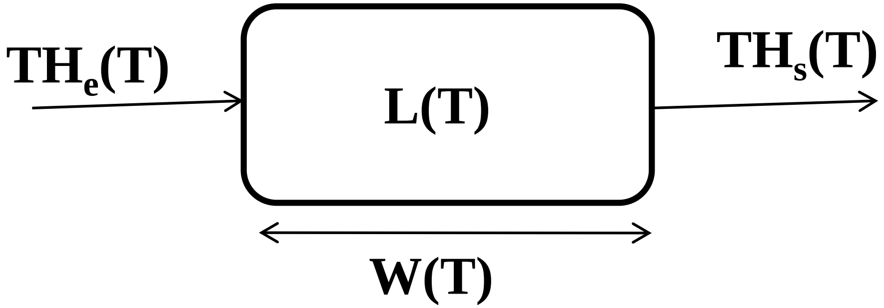
Little's Law

- Assuming a queuing environment to be operating in a stable steady state where all initial transients have vanished, the key parameters characterizing the system are:
  - λ – the mean steady state consumer arrival
  - N – the average no. of customers in the system
  - T – the mean time spent by each customer in the system

which gives

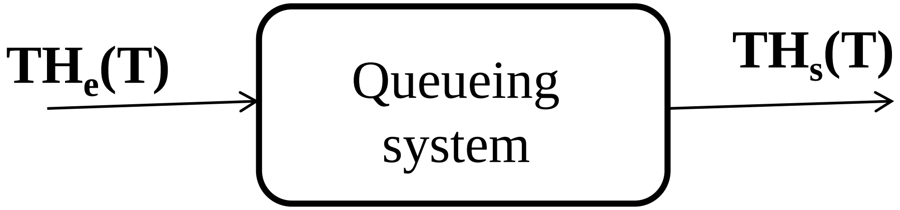
$$N = \lambda T$$

Some transient performances



- A(T) : **number of customers arrived** from 0 to T
- D(T) : **number of departures** between 0 to T
- $TH_e(T) = A(T)/T$: **average arrival rate** between 0 to T
- $TH_s(T) = D(T)/T$: **average departure rate** between 0 to T
- L(T) : **average number of customers** between 0 to T
- W_k: **sojourn time** of k-th customer in the system
- $W(T) = \frac{1}{A(T)} \sum_{k=1}^{A(T)} W_k$ **average sojourn time** between 0 to T

Stability of the queueing system



Definition : A queueing system is said **stable** if the number of customers in the system remains finite.

Implication of the stability:

$$\lim_{T\to\infty} TH_e(T) = \lim_{T\to\infty} TH_s(T)$$

$$\lim_{T\to\infty} \frac{D(T)}{A(T)} = 1$$

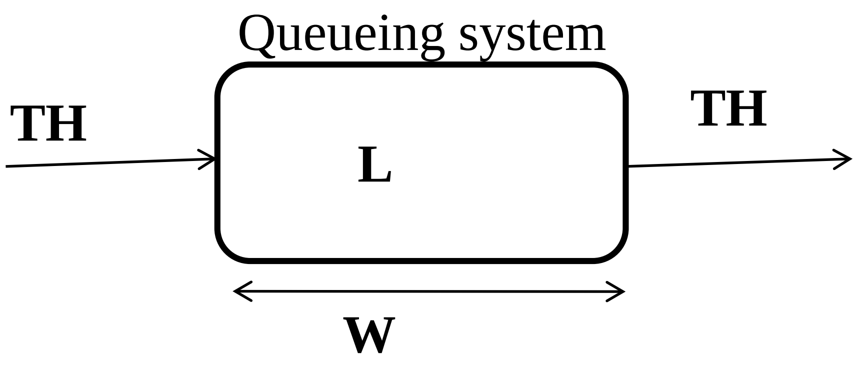
Little's law

For a stable and ergodic queueing system,

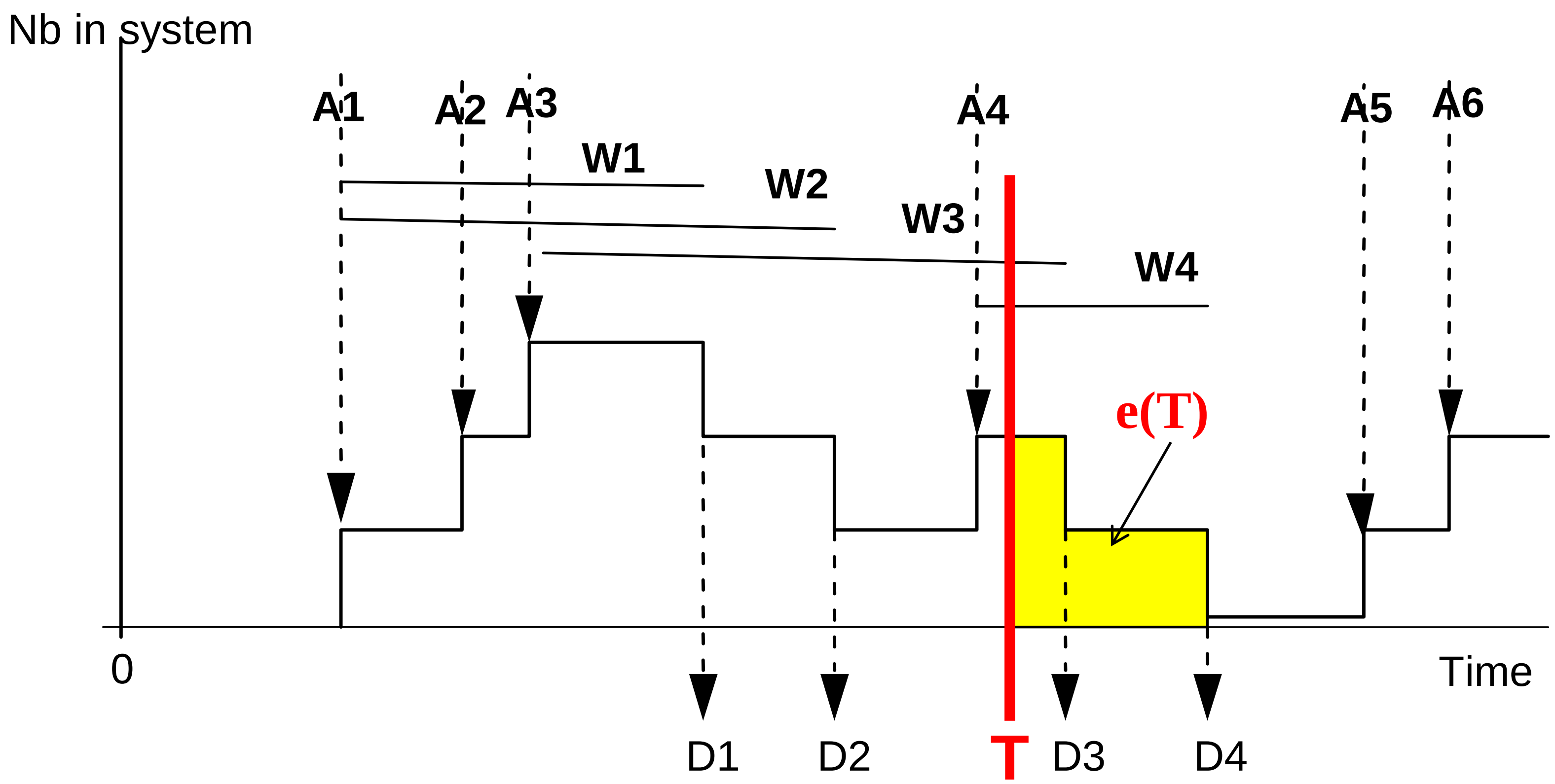
$$L = TH \times W$$

where

- L : average number of customers in the system
- W : average response time
- TH : average throughput rate



Proof



Proof

$$R(T)TH(T) = \left(\frac{1}{A(T)}\sum_{k=1}^{A(T)} R_k\right)\left(\frac{D(T)}{T}\right) = \left(\frac{D(T)}{A(T)}\right)\left(\frac{1}{T}\sum_{k=1}^{A(T)} R_k\right)$$

$$\frac{1}{T}\sum_{k=1}^{A(T)} R_k = \frac{1}{T}\sum_{k=1}^{A(T)} \int_{t=0}^{T} 1(k \text{ still there at } t)\,dt + \frac{1}{T}\sum_{k=A(T)-N(T)+1}^{A(T)} r_k(T)$$

$$= Q(T) + \frac{1}{T} e(T)$$

where N(T) is the number of customers at time T, e(T) **total remaining system time** of customers present at time T.

Letting T go to infinity, the stability implies the proof.