Chapter 1 Preliminary

Preliminaries

- Applied Probability and Performance Modeling
  - Prototype
  - System Simulation
  - Probabilistic Model
- Introduction to Stochastic Processes
  - Random Variable (R.V.)
  - Stochastic Process
- Probability and Expectations
  - Expectation
  - Generating Functions for Discrete R.V.s
  - Laplace Transforms for Continuous R.V.s
  - Moment Generating Functions

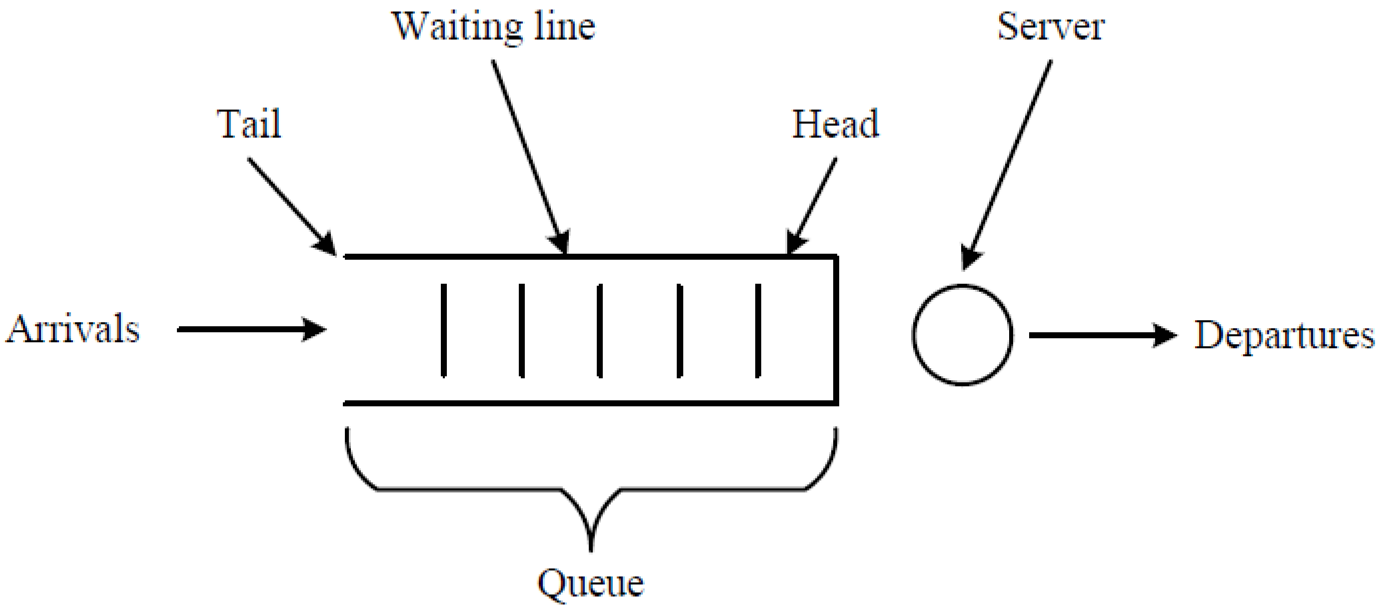
Preliminaries

- Probability Inequalities
  - Markov's Inequality (mean)
  - Chebyshev's Inequality (mean and variance)
  - Chernoff's Bound (moment generating function)
  - Jensen's Inequality
- Limit Theorems
  - Strong Law of Large Numbers
  - Weak Law of Large Numbers
  - Central Limit Theorem

Applied Probability and Performance Modeling

- Prototyping
  - complex and expensive
  - provides information on absolute performance measures but little on relative performance of different designs
- System Simulation
  - large amount of execution time
  - could provide both absolute and relative performance depending on the level of detail that is modeled
- Probabilistic Model
  - mathematically intractable or unsolvable
  - provide great insight into relative performance but, often, are not accurate representations of absolute performance

A Single Server Queue



- Arrivals: Poisson process, renewal process, etc.
- Queue length: Markov process, semi-Markov process, etc.
- …

Random Variable

- A "***random variable***" is a real-valued function whose domain is a sample space.
- **Example**: Suppose that our experiment consists of tossing 3 fair coins. If we let $\tilde{y}$ denote the number of heads appearing, then $\tilde{y}$ is a random variable taking on one of the values 0, 1, 2, 3 with respective probabilities

$$
\begin{aligned}
P\{\tilde{y}=0\} &= P\{(T,T,T)\} = \frac{1}{8} \\
P\{\tilde{y}=1\} &= P\{(T,T,H),(T,H,T),(H,T,T)\} = \frac{3}{8} \\
P\{\tilde{y}=2\} &= P\{(T,H,H),(H,T,H),(H,H,T)\} = \frac{3}{8} \\
P\{\tilde{y}=3\} &= P\{(H,H,H)\} = \frac{1}{8}
\end{aligned}
$$

Random Variable

- A random variable $\tilde{x}$ is said to be "***discrete***" if it can take on only a finite number–or a countable infinity – of possible values *x*.
- A random variable $\tilde{x}$ is said to be "***continuous***" if there exists a nonnegative function *f*, defined for all real $x \in (-\infty, \infty)$, having the property that for any set *B* of real numbers

$$P\{\tilde{x} \in B\} = \int_B f(x)dx$$

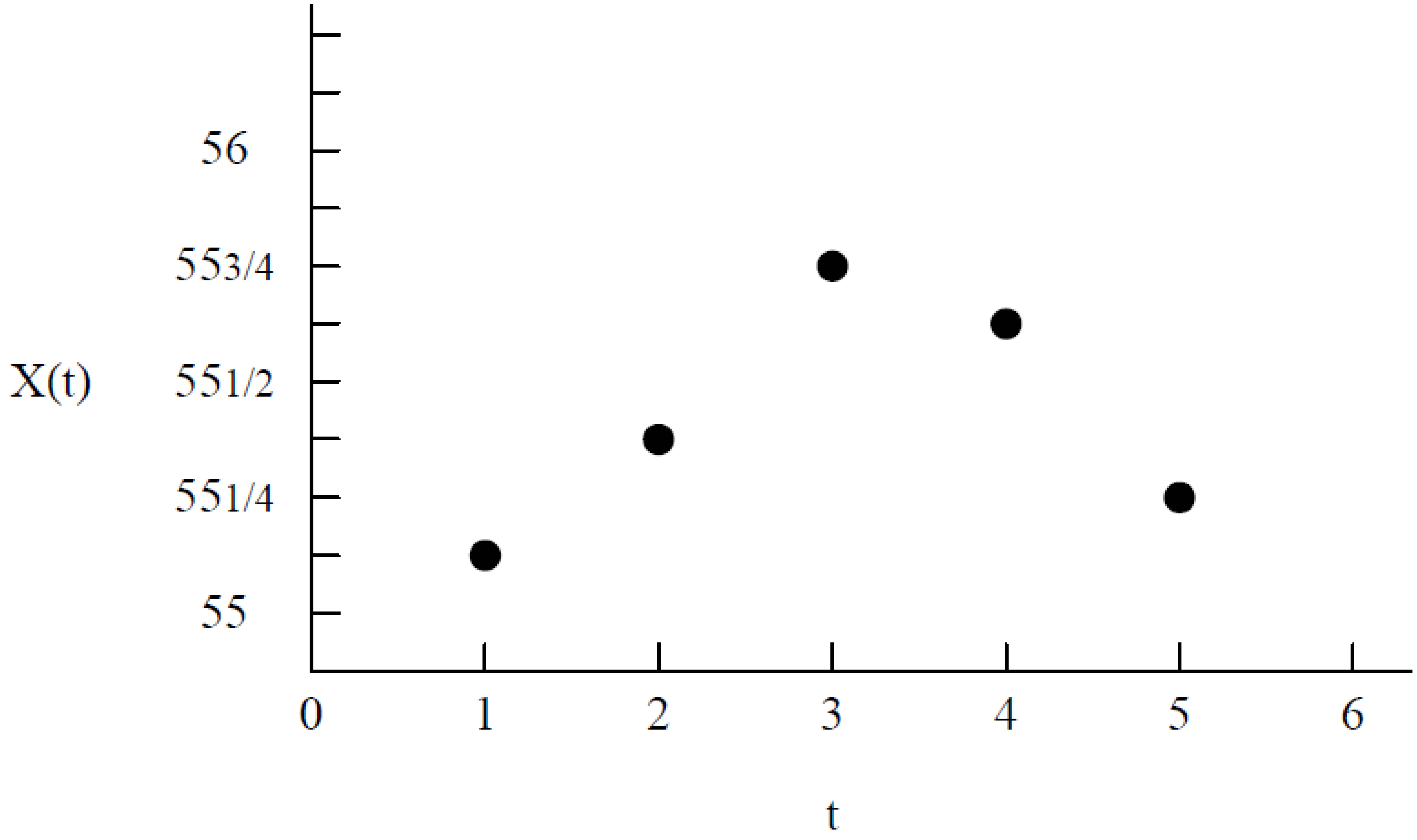
Stochastic Process

- A "***stochastic process***" $X = \{\tilde{x}(t), t \in T\}$ is a collection of random variables. That is, for each $t \in T$, $\tilde{x}(t)$ is a random variable.
- The index *t* is often interpreted as "*time*" and, as a result, we refer to $\tilde{x}(t)$ as the "*state*" of the process at time *t*.
- When the index set *T* of the process *X* is
  - a countable set → *X* is a *discrete-time* process
  - an interval of the real line → *X* is a *continuous-time* process
- When the state space *S* of the process *X* is
  - a countable set → *X* has a *discrete state space*
  - an interval of the real line → *X* has a *continuous state space*

Stochastic Process

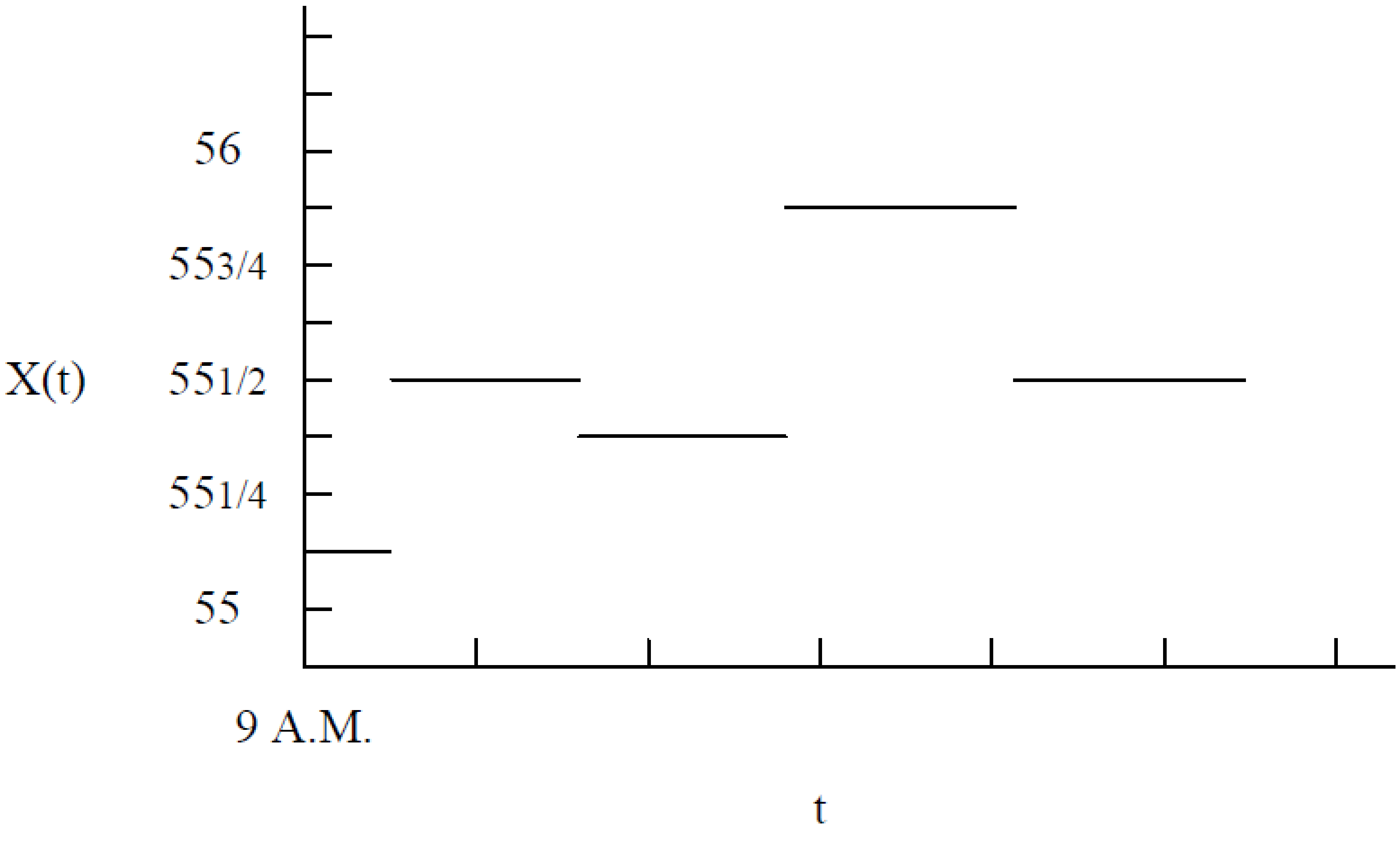
- Four types of stochastic processes
  - discrete time and discrete state space
  - continuous time and discrete state space
  - discrete time and continuous state space
  - continuous time and continuous state space

Discrete Time with Discrete State Space



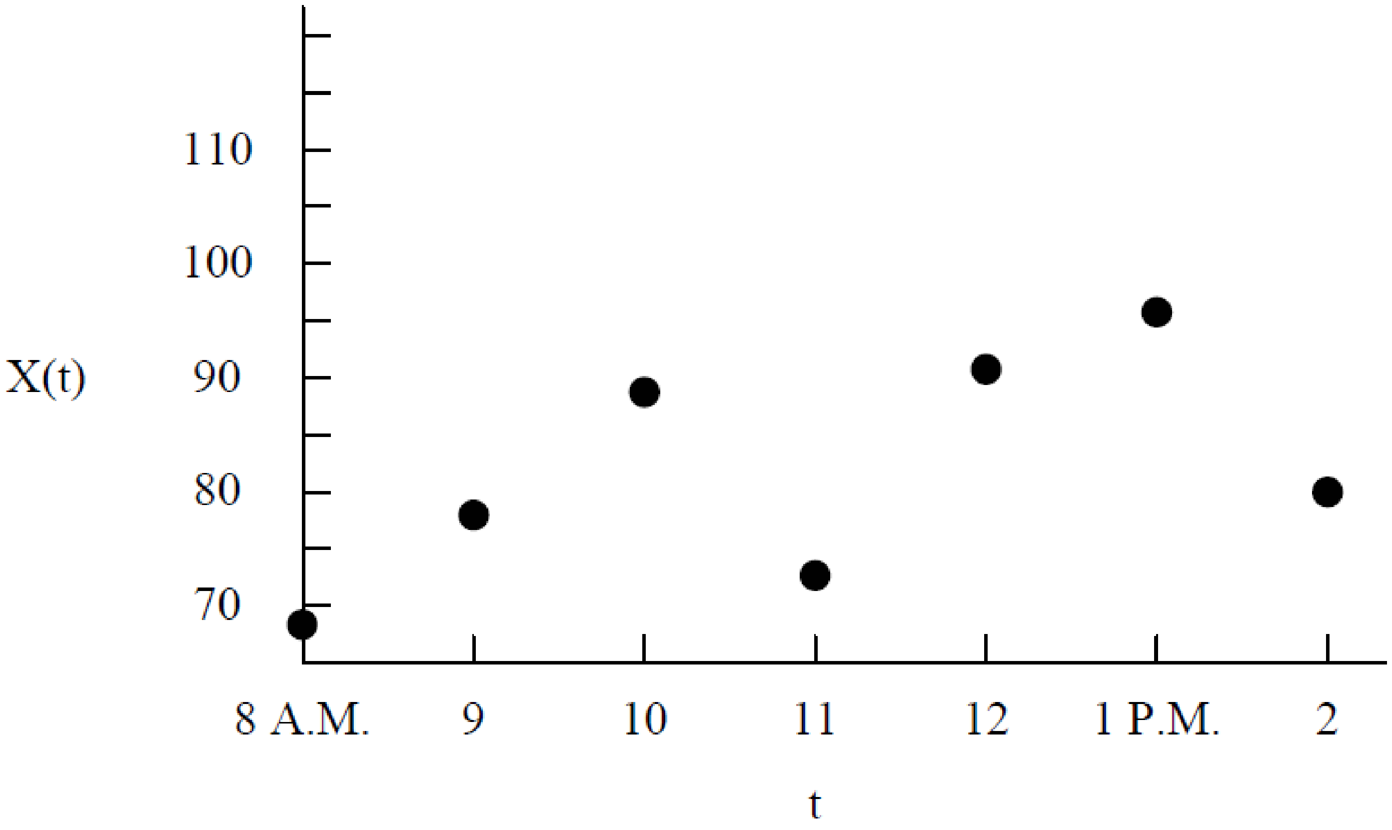
X(t) = closing price of an IBM stock on day t

Continuous Time with Discrete State Space



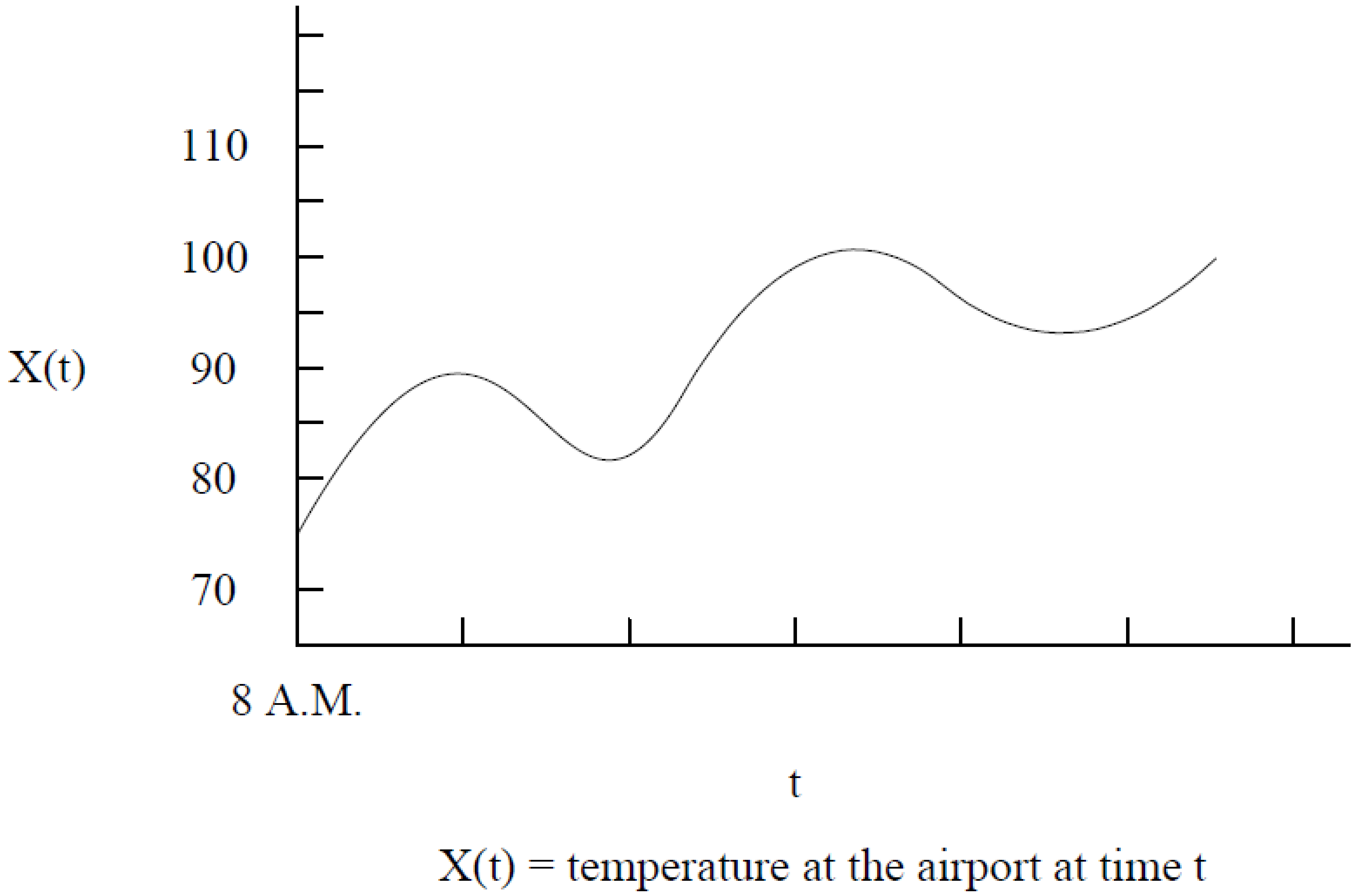
X(t) = price of an IBM stock at time t on a given day

Discrete Time with Continuous State Space



X(t) = temperature at the airport at time t

Continuous Time with Continuous State Space



Two Structural Properties of stochastic processes

- **Independent increment:** if for all $t_0 < t_1 < t_2 < \ldots < t_n$ in the process
  - $X = \{\tilde{x}(t), t \in T\}$, random variables
  - $\tilde{x}(t_1) - \tilde{x}(t_0)$, $\tilde{x}(t_2) - \tilde{x}(t_1)$, … $\tilde{x}(t_n) - \tilde{x}(t_{n-1})$ are independent,

  →the magnitudes of state change over non-overlapping time intervals are mutually independent

- **Stationary increment:** if the random variable $\tilde{x}(t+s)$ - $\tilde{x}(t)$ has the same probability distribution for all t and any $s > 0$,

  → the probability distribution governing the magnitude of state change depends only on the difference in the lengths of the time indices and is independent of the time origin used for the indexing variable

$$\Downarrow$$

$$X = \{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots, \tilde{x}_\infty\}$$

limiting behavior of the stochastic process

Two Structural Properties of stochastic processes

- both independent and stationary increments,
- neither independent nor stationary increments,
- independent but not stationary increments, and
- stationary but not independent increments.

Expectations by Conditioning

- Denote by $E\left[\tilde{x}\middle|\tilde{y}\right]$that function of the random variable $\tilde{y}$ whose value at $\tilde{y}$ = *y* is $E[\tilde{x}|\tilde{y} = y]$.

  $\rightarrow E[\tilde{x}] = E\left[E\left[\tilde{x}\middle|\tilde{y}\right]\right]$

- If$\tilde{y}$ is a discrete random variable, then
  - $E[\tilde{x}]=\sum_y E[\tilde{x}|\tilde{y} = y]P\{\tilde{y} = y\}$
- If $\tilde{y}$ is continuous with density $f_{\tilde{y}}(y)$, then
  - $E[\tilde{x}]=\int_{-\infty}^{\infty} E[\tilde{x}|\tilde{y} = y]\, f_{\tilde{y}}(y)\, dy$

Expectations by Complementary Distribution

- For any non-negative random variable $\tilde{x}$

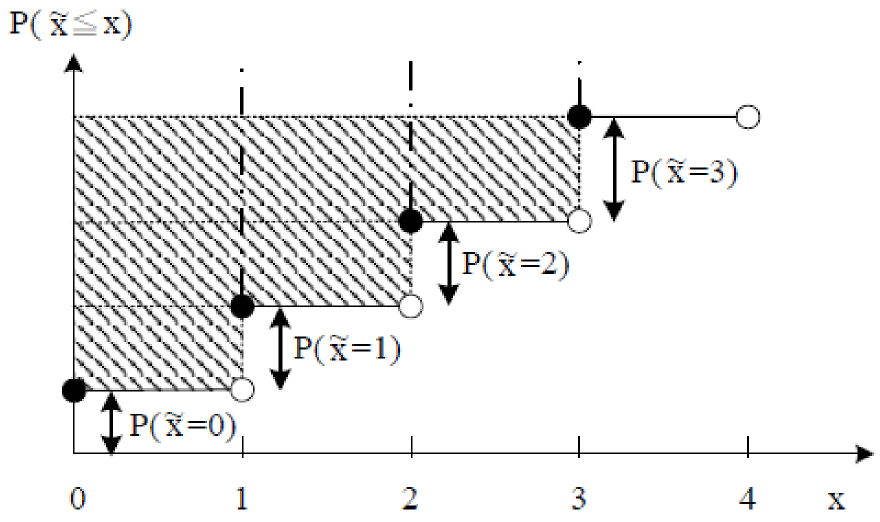
$$E[\tilde{x}] = \sum_{k=0}^{\infty} p(\tilde{x} > k) \qquad \text{discrete}$$

$$E[\tilde{x}] = \int_0^{\infty} [1 - F_{\tilde{x}}(x)]dx \qquad \text{continuous}$$

Expectations by Complementary Distribution

- Discrete case:

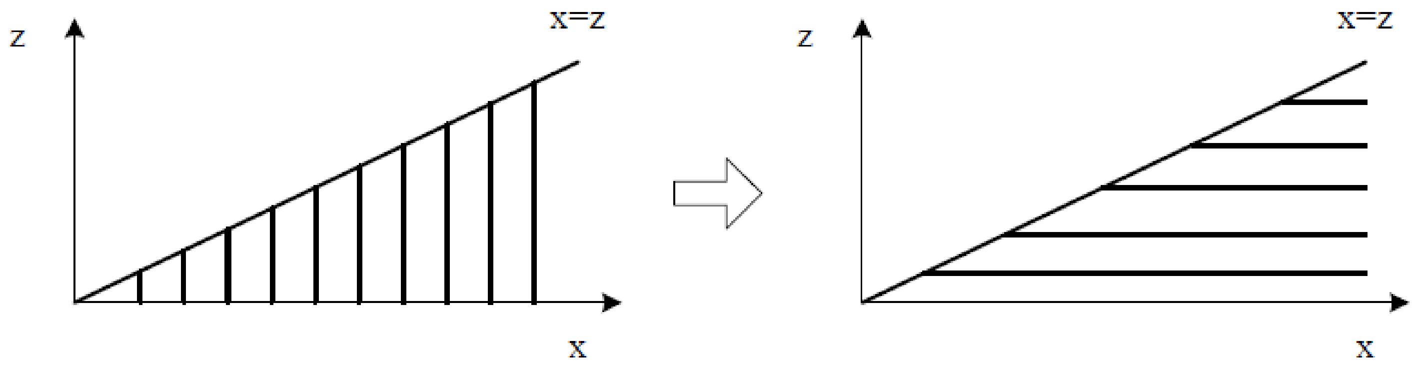
$$
\begin{aligned}
E[\tilde{x}] &= 0 \cdot P(\tilde{x}=0) + 1 \cdot P(\tilde{x}=1) + 2 \cdot P(\tilde{x}=2) + \ldots \quad \text{(horizontal sum)} \\
&= [1 - P(\tilde{x}<1)] + [1 - P(\tilde{x}<2)] + \ldots \quad \text{(vertical sum)} \\
&= P(\tilde{x} \geq 1) + P(\tilde{x} \geq 2) + \ldots \\
&= \sum_{k=1}^{\infty} P(\tilde{x} \geq k) \qquad \left(or \sum_{k=0}^{\infty} P(\tilde{x} > k)\right)
\end{aligned}
$$



Expectations by Complementary Distribution

- Continuous case:

$$
\begin{aligned}
E[\tilde{x}] &= \int_0^{\infty} x \cdot f_{\tilde{x}}(x)dx \\
&= \int_0^{\infty} \left(\int_0^{x} dz \right) \cdot f_{\tilde{x}}(x)dx \\
&= \int_0^{\infty} \left[\int_z^{\infty} f_{\tilde{x}}(x)dx \right] \cdot dz \\
&= \int_0^{\infty} [1 - F_{\tilde{x}}(z)]dz
\end{aligned}
$$



Compound Random Variable

$\tilde{S}_{\tilde{n}} = \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 + \ldots + \tilde{x}_{\tilde{n}}$, where $\tilde{n} \geq 1$ and $\tilde{x}_i$ are i.i.d. random variables.

$\Rightarrow E[\tilde{S}_{\tilde{n}}] =? \; Var[\tilde{S}_{\tilde{n}}] =?$

$$\begin{aligned}
E[\tilde{S}_{\tilde{n}}] &= E[E[\tilde{S}_{\tilde{n}}|\tilde{n}]] \\
&= \sum_{n=1}^{\infty} E[\tilde{S}_{\tilde{n}}|\tilde{n} = n] \cdot P(\tilde{n} = n) \\
&= \sum_{n=1}^{\infty} E[\tilde{x}_1 + \tilde{x}_2 + \ldots + \tilde{x}_n] \cdot P(\tilde{n} = n) \\
&= \sum_{n=1}^{\infty} n \cdot E[\tilde{x}_1] \cdot P(\tilde{n} = n) \\
&= \boxed{E[\tilde{n}] \cdot E[\tilde{x}_1]}
\end{aligned}$$

Compound Random Variable

Since $Var[\tilde{x}] = E[Var[\tilde{x}|\tilde{y}]] + Var[E[\tilde{x}|\tilde{y}]]$, we have

$$\begin{aligned} Var[\tilde{S}_{\tilde{n}}] &= E[Var[\tilde{S}_{\tilde{n}}|\tilde{n}]] + Var[E[\tilde{S}_{\tilde{n}}|\tilde{n}]] \\ &= E[\tilde{n}Var[\tilde{x}_1]] + Var[\tilde{n}E[\tilde{x}_1]] \\ &= Var[\tilde{x}_1]E[\tilde{n}] + E^2[\tilde{x}_1]Var[\tilde{n}] \end{aligned}$$

Chapter 2 Poisson Processes

Outline

- Introduction to Poisson Processes
- Properties of Poisson processes
  - Inter-arrival time distribution
  - Waiting time distribution
  - Superposition and decomposition
- Non-homogeneous Poisson processes (relaxing *stationary*)
- Compound Poisson processes (relaxing *single arrival*)
- Modulated Poisson processes (relaxing *independent*)
- Poisson Arrival See Time Average (PASTA)

兩個Poisson processes 相加

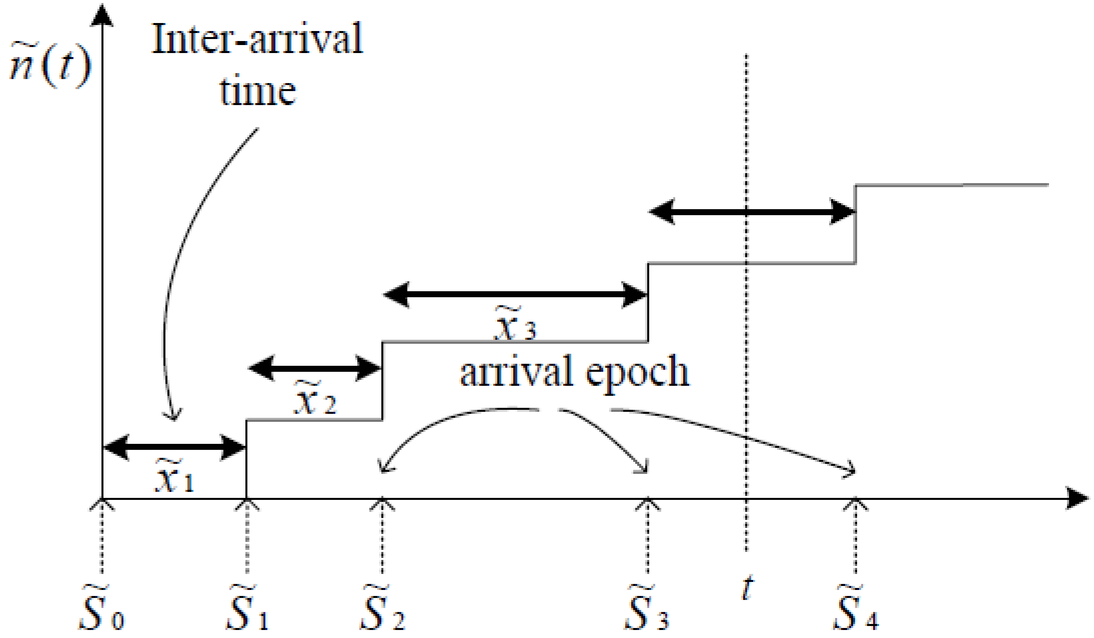
Introduction

(i) n_{th} arrival epoch $\tilde{S}_n$ is

$$\begin{aligned}\tilde{S}_n &= \tilde{x}_1 + \tilde{x}_2 + \ldots + \tilde{x}_n = \textstyle\sum_{i=1}^{n} \tilde{x}_i \\ \tilde{S}_0 &= 0\end{aligned}$$

(ii) Number of arrivals at time t is: $\tilde{n}(t)$. Notice that:

$$\{\tilde{n}(t) \geq n\} \overset{iff}{\Leftrightarrow} \{\tilde{S}_n \leq t\},\ \{\tilde{n}(t) = n\} \overset{iff}{\Leftrightarrow} \{\tilde{S}_n \leq t \text{ and } \tilde{S}_{n+1} > t\}$$



Introduction

Arrival Process: $X = \{\tilde{x}_i,\ i = 1, 2, \ldots\}$; $\tilde{x}_i$'s can be any

$S = \{\tilde{S}_i,\ i = 0, 1, 2, \ldots\}$; $\tilde{S}_i$'s can be any

$N = \{\tilde{n}(t),\ t \geq 0\}$; $\longrightarrow$ called arrival process

Renewal Process: $X = \{\tilde{x}_i,\ i = 1, 2, \ldots\}$; $\tilde{x}_i$'s are i.i.d.

$S = \{\tilde{S}_i,\ i = 0, 1, 2, \ldots\}$; $\tilde{S}_i$'s are general distributed

$N = \{\tilde{n}(t),\ t \geq 0\}$; $\longrightarrow$ called renewal process

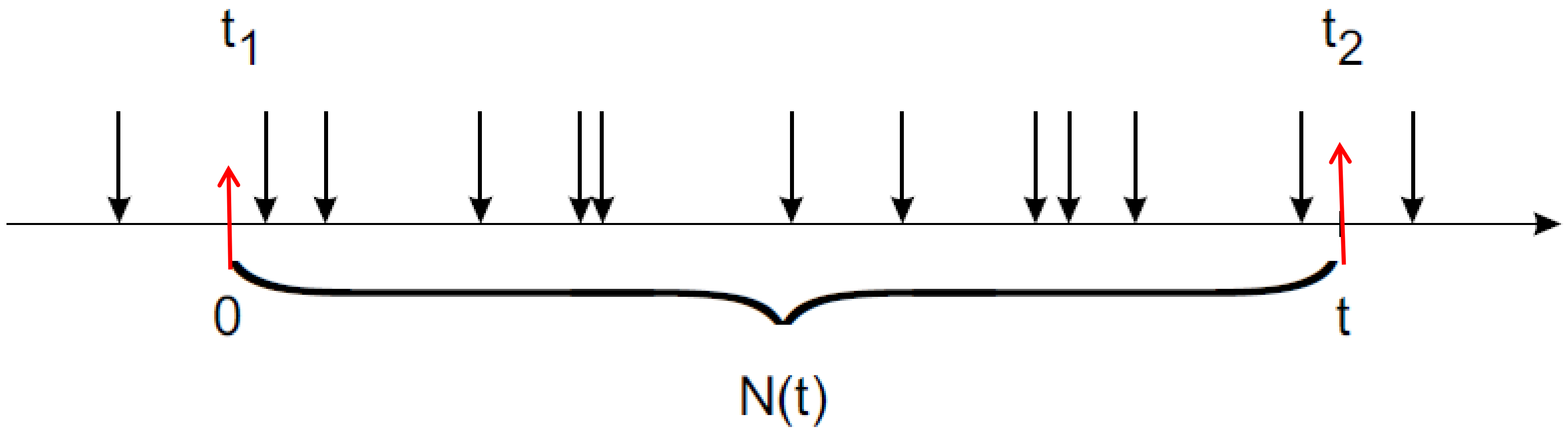
Poisson Process: $X = \{\tilde{x}_i,\ i = 1, 2, \ldots\}$; $\tilde{x}_i$'s are iid exponential distributed

$S = \{\tilde{S}_i,\ i = 0, 1, 2, \ldots\}$; $\tilde{S}_i$'s are Erlang distributed

$N = \{\tilde{n}(t),\ t \geq 0\}$; $\longrightarrow$ called Poisson process

Poisson process

- Poisson process is one of the most important models used in queueing theory.
  - Often the arrival process of customers can be described by a Poisson process.
  - In teletraffic theory the “customers” may be calls or packets.
  - Poisson process is a viable model when the calls or packets originate from a large population of independent users.
- In the following, it is instructive to think that the Poisson process we consider represents discrete arrivals (of e.g. calls or packets).



Poisson Arrival Model

- A Poisson process is a sequence of events "randomly spaced in time"
- For example, customers arriving at a bank and Geiger counter clicks are similar to packets arriving at a buffer
- The rate λ of a Poisson process is the average number of events per unit time (over a long time)

Poisson process

- Mathematically the process is described by the so called *counter process* N_t or *N(t)*.
- The counter tells the number of arrivals that have occurred in the interval (0, t) or, more generally, in the interval (t1, t2).

$$\begin{cases} N(t) = \text{number of arrivals in the interval } (0,t) & \text{(the stochastic process we consider)} \\ N(t_1,t_2) = \text{number of arrival in the interval } (t_1,t_2) & \text{(the increment process } N(t_2)-N(t_1)) \end{cases}$$

- A Poisson process can be characterized in different ways:
  - Process of independent increments
  - Pure birth process
    - The arrival intensity (mean arrival rate; probability of arrival per time unit)
  - The “most random” process with a given intensity λ

Properties of a Poisson Process

- Properties of a Poisson process
  - For a time interval [0, t] , the probability of *n* arrivals in *t* units of time is

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

  - For two disjoint (non overlapping) intervals (t1, t2) and (t3, t4), (i.e. , t1 < t2 < t3 < t4), the number of arrivals in (t1, t2) is *independent* of arrivals in (t3, t4)

Counting Processes

- A stochastic process *N* = {$\tilde{n}(t)$, t ≥ 0} is said to be a *counting process* if $\tilde{n}(t)$ represents the total number of "events" that have occurred up to time t.
- From the definition, we see that for a counting process $\tilde{n}(t)$ must satisfy:

1. $\tilde{n}(t) \geq 0$.
2. $\tilde{n}(t)$ is integer valued.
3. If $s < t$, then $\tilde{n}(s) \leq \tilde{n}(t)$.
4. For $s < t$, $\tilde{n}(t) - \tilde{n}(s)$ equals the number of events that have occurred in the interval $(s, t]$.

Definition 1: Poisson Processes

- The counting process $N = \{\tilde{n}(t)\ , t \geq 0\}$ is a *Poisson process* with rate λ ($\lambda > 0$), if:

1. $\tilde{n}(0) = 0$

是指任兩段不重疊的區間內的事件發生次數互不相干

2. Independent increments relaxed $\Rightarrow$ Modulated Poisson Process

$$P[\tilde{n}(t) - \tilde{n}(s) = k_1 | \tilde{n}(r) = k_2,\ r \leq s < t] = P[\tilde{n}(t) - \tilde{n}(s) = k_1]$$

3. Stationary increments relaxed $\Rightarrow$ Non-homogeneous Poisson Process

$$P[\tilde{n}(t+s) - \tilde{n}(t) = k] = P[\tilde{n}(l+s) - \tilde{n}(l) = k]$$

是指某個區間內事件發生次數的機率分配只跟那段區間的長度有關。

4. Single arrival relaxed $\Rightarrow$ Compound Poisson Process

$$P[\tilde{n}(h) = 1] = \lambda h + o(h)$$
$$P[\tilde{n}(h) \geq 2] = o(h)$$

在極短或很小的區域，發生超過一次事件的情況 微乎其微，亦即將時間或區域細分至極小單位， 則事件不是只出現一次，就是不出現。

Definition 2: Poisson Processes

- The counting process $N = \{\tilde{n}(t), t \geq 0\}$ is a *Poisson process* with rate λ ($\lambda > 0$), if:

1. $\tilde{n}(0) = 0$
2. Independent increments
3. The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geq 0$

$$P[\tilde{n}(t+s) - \tilde{n}(s) = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

Theorem: Definitions 1 and 2 are equivalent.

- **Proof.** We show that Definition 1 implies Definition 2. To start, fix $u \geq 0$ and let

$$g(t) = E[e^{-u\tilde{n}(t)}]$$

We derive a differential equation for $g(t)$ as follows:

$$\begin{aligned} g(t+h) &= E[e^{-u\tilde{n}(t+h)}] \\ &= E\left\{e^{-u\tilde{n}(t)}e^{-u[\tilde{n}(t+h)-\tilde{n}(t)]}\right\} \\ &= E\left[e^{-u\tilde{n}(t)}\right]E\left\{e^{-u[\tilde{n}(t+h)-\tilde{n}(t)]}\right\} \quad \text{by independent increments} \\ &= g(t)E\left[e^{-u\tilde{n}(h)}\right] \quad \text{by stationary increments} \qquad (1) \end{aligned}$$

Theorem: Definitions 1 and 2 are equivalent.

Conditioning on whether $\tilde{n}(t) = 0$ or $\tilde{n}(t) = 1$ or $\tilde{n}(t) \geq 2$ yields

$$\begin{aligned} E\left[e^{-u\tilde{n}(h)}\right] &= 1 - \lambda h + o(h) + e^{-u}(\lambda h + o(h)) + o(h) \\ &= 1 - \lambda h + e^{-u}\lambda h + o(h) \qquad (2) \end{aligned}$$

From (1) and (2), we obtain that

$$g(t+h) = g(t)(1 - \lambda h + e^{-u}\lambda h) + o(h)$$

implying that

differential (微分) ←

$$\frac{g(t+h) - g(t)}{h} = g(t)\lambda(e^{-u} - 1) + \frac{o(h)}{h}$$

Theorem: Definitions 1 and 2 are equivalent.

Letting $h \to 0$ gives

$$g'(t) = g(t)\lambda(e^{-u} - 1)$$

or, equivalently,

$$\frac{g'(t)}{g(t)} = \lambda(e^{-u} - 1)$$

Integrating, and using $g(0) = 1$, shows that

$$\log(g(t)) = \lambda t(e^{-u} - 1)$$

or

$$g(t) = e^{\lambda t(e^{-u}-1)} \quad \to \quad \text{the Laplace transform of a Poisson r. v.}$$

Since $g(t)$ is also the Laplace transform of $\tilde{n}(t)$, $\tilde{n}(t)$ is a Poisson r. v.

Interarrival Times of Poisson Process

- Interarrival times of a Poisson process
  - We pick an arbitrary starting point t0 in time. Let T1 be the time until the next arrival. We have

    $\int e^x dx = e^x + C$

    $P(T1 > t0) = P0(t) = e^{-\lambda t}$
  - Thus the cumulative distribution function of T1 is given by

    $FT1(t) = P(T1 \leq t) = 1 – e^{-\lambda t}$
  - The pdf of T1 is given by $f(x) = \frac{dF_X(x)}{dx}$

    $fT1(t) = \lambda e^{-\lambda t}$
  - Therefore, T1 has an exponential distribution with mean rate λ

The Inter-Arrival Time Distribution

- **Theorem.** Poisson Processes have exponential inter-arrival time distribution, i.e., $\{\widetilde{x_n}, n = 1, 2, \ldots\}$ are i.i.d and exponentially distributed with parameter λ (i.e., mean inter-arrival time = $1/\lambda$).

Proof.

$$\tilde{x}_1 \quad : \quad P(\tilde{x}_1 > t) = P(\tilde{n}(t) = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$\therefore \tilde{x}_1 \sim e(t; \lambda)$$

$$\tilde{x}_2 \quad : \quad P(\tilde{x}_2 > t | \tilde{x}_1 = s)$$

$$= P\{0 \text{ arrivals in } (s, s+t] | \tilde{x}_1 = s\}$$

$$= P\{0 \text{ arrivals in } (s, s+t]\} (\text{by independent increment})$$

$$= P\{0 \text{ arrivals in } (0, t]\} (\text{by stationary increment})$$

$$= e^{-\lambda t} \quad \therefore \tilde{x}_2 \text{ is independent of } \tilde{x}_1 \text{ and } \tilde{x}_2 \sim exp(t; \lambda).$$

$\Rightarrow$ The procedure repeats for the rest of $\tilde{x}_i$'s.

The Arrival Time Distribution of the *n*th Event

- **Theorem.** The arrival time of the n_{th} event, $\widetilde{S_n}$(also called the waiting time until the *n*th event), is *Erlang* distributed with parameter (*n, λ*).

Proof. Method 1 :

$$\because \quad P[\tilde{S}_n \le t] = P[\tilde{n}(t) \ge n] = \sum_{k=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

$$\therefore \quad f_{\tilde{S}_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!} \quad \text{(exercise)}$$

Method 2 :

$$\begin{aligned} f_{\tilde{S}_n}(t)dt &= dF_{\tilde{S}_n}(t) = P[t < \tilde{S}_n < t + dt] \\ &= P\{n-1 \text{ arrivals in } (0,t] \text{ and } 1 \text{ arrival in } (t, t+dt)\} + o(dt) \\ &= P[\tilde{n}(t) = n-1 \text{ and } 1 \text{ arrival in } (t, t+dt)] + o(dt) \\ &= P[\tilde{n}(t) = n-1]P[1 \text{ arrival in } (t, t+dt)] + o(dt)(why?) \end{aligned}$$

independent increments

The Arrival Time Distribution of the *n*th Event

$$= \quad \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}\lambda dt + o(dt)$$

$$\therefore \quad \lim_{dt\to 0} \frac{f_{\tilde{S}_n}(t)dt}{dt} = f_{\tilde{S}_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}$$

Conditional Distribution of the Arrival Times

- **Theorem.** Given that $\tilde{n}(t)$ = *n*, the *n* arrival times $\widetilde{S_1}, \widetilde{S_2}, \ldots, \widetilde{S_n}$ have the same distribution as the order statistics corresponding to *n* i.i.d. uniformly distributed random variables from (0, *t*).

Order Statistics. Let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ be n i.i.d. continuous random variables having common pdf f. Define $\tilde{x}_{(k)}$ as the k_{th} smallest value among all $\tilde{x}_i$'s, i.e., $\tilde{x}_{(1)} \leq \tilde{x}_{(2)} \leq \tilde{x}_{(3)} \leq \ldots \leq \tilde{x}_{(n)}$, then $\tilde{x}_{(1)}, \ldots, \tilde{x}_{(n)}$ are known as the "order statistics" corresponding to random variables $\tilde{x}_1, \ldots, \tilde{x}_n$. We have that the joint pdf of $\tilde{x}_{(1)}, \tilde{x}_{(2)}, \ldots, \tilde{x}_{(n)}$ is

$$f_{\tilde{x}_{(1)}, \tilde{x}_{(2)}, \ldots, \tilde{x}_{(n)}}(x_1, x_2, \ldots, x_n) = n! f(x_1) f(x_2) \ldots f(x_n),$$

where $x_1 < x_2 < \ldots < x_n$ (check the textbook [Ross]).

Conditional Distribution of the Arrival Times

Proof. Let $0 < t_1 < t_2 < \ldots < t_{n+1} = t$ and let h_i be small enough so that $t_i + h_i < t_{i+1}$, $i = 1, \ldots, n$.

$$\because \quad P[t_i < \tilde{S}_i < t_i + h_i,\ i = 1, \ldots, n | \tilde{n}(t) = n]$$

$$= \frac{P\begin{pmatrix} \text{exactly one arrival in each } [t_i, t_i + h_i] \\ i = 1, 2, \ldots, n, \text{ and no arrival elsewhere in } [0, t] \end{pmatrix}}{P[\tilde{n}(t) = n]}$$

$$= \frac{(e^{-\lambda h_1}\lambda h_1)(e^{-\lambda h_2}\lambda h_2)\ldots(e^{-\lambda h_n}\lambda h_n)(e^{-\lambda(t-h_1-h_2\ldots-h_n)})}{e^{-\lambda t}(\lambda t)^n/n!}$$

$$= \frac{n!(h_1 h_2 h_3 \ldots h_n)}{t^n}$$

$$\therefore \quad \frac{P[t_i < \tilde{S}_i < t_i + h_i,\ i = 1, \ldots, n | \tilde{n}(t) = n]}{h_1 h_2 \ldots h_n} = \frac{n!}{t^n}$$

Conditional Distribution of the Arrival Times

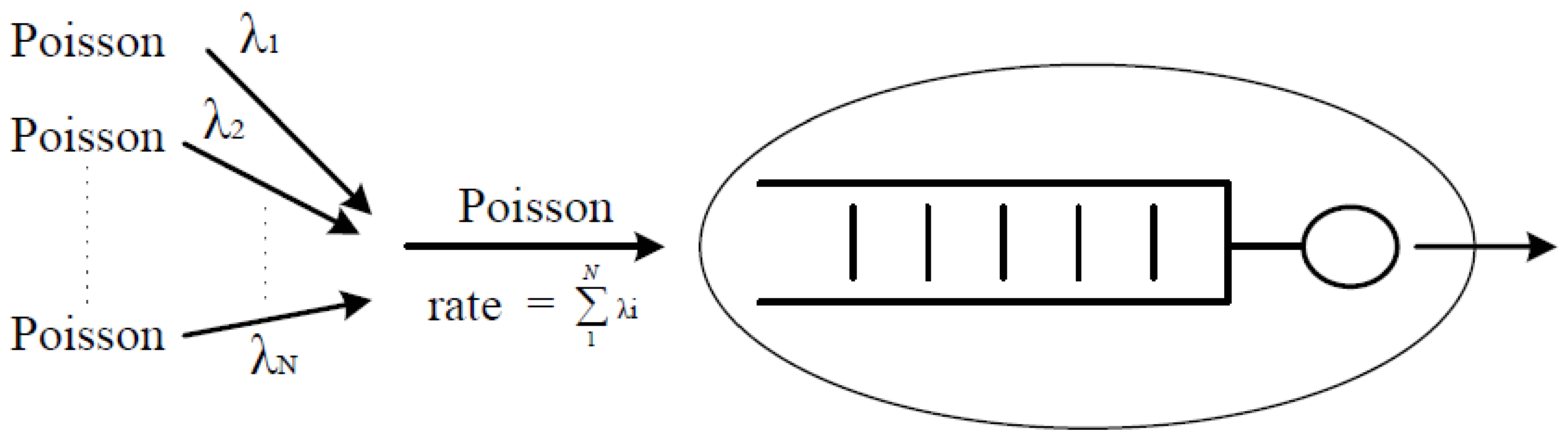
Taking $\lim_{h_i \to 0, i=1,\ldots,n} (\qquad)$, then

$$f_{\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_n | \tilde{n}(t)}(t_1, t_2, \ldots, t_n | n) = \frac{n!}{t^n},\ 0 < t_1 < t_2 < \ldots < t_n.$$

Superposition of Independent Poisson Processes

- **Theorem.** Superposition of independent Poisson Processes

$(\lambda_i, i = 1, \ldots, N)$, is also a Poisson process with rate $\sum_1^N \lambda_i$.



Decomposition of a Poisson Process

Theorem.

- Given a Poisson process $N = \{\tilde{n}(t), t \geq 0\}$;
- If $\tilde{n}_i(t)$ represents the number of type-i events that occur by time $t, i = 1, 2$;
- Arrival occurring at time s is a type-1 arrival with probability $p(s)$, and type-2 arrival with probability $1 - p(s)$

$\Downarrow$then

- $\tilde{n}_1, \tilde{n}_2$ are independent,
- $\tilde{n}_1(t) \sim P(k; \lambda t p)$, and
- $\tilde{n}_2(t) \sim P(k; \lambda t(1-p))$, where $p = \frac{1}{t}\int_0^t p(s)ds$

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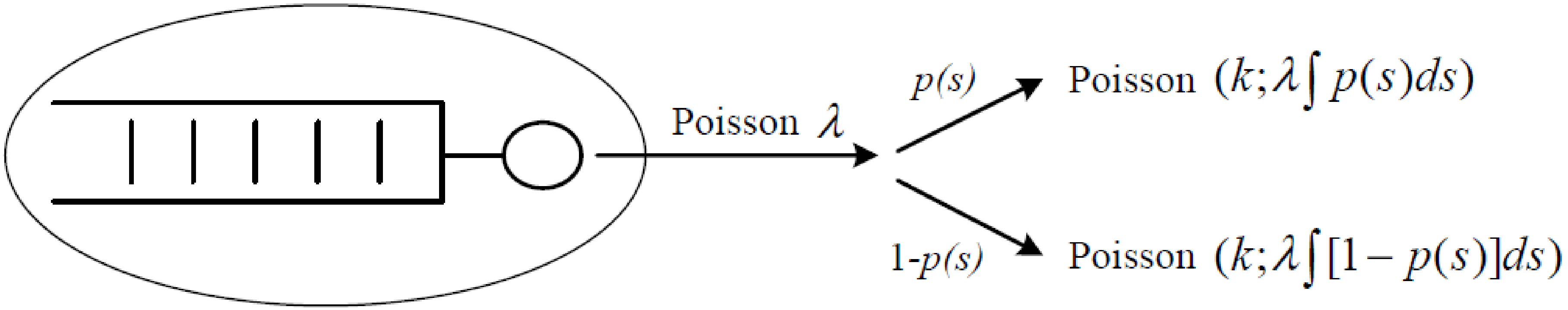
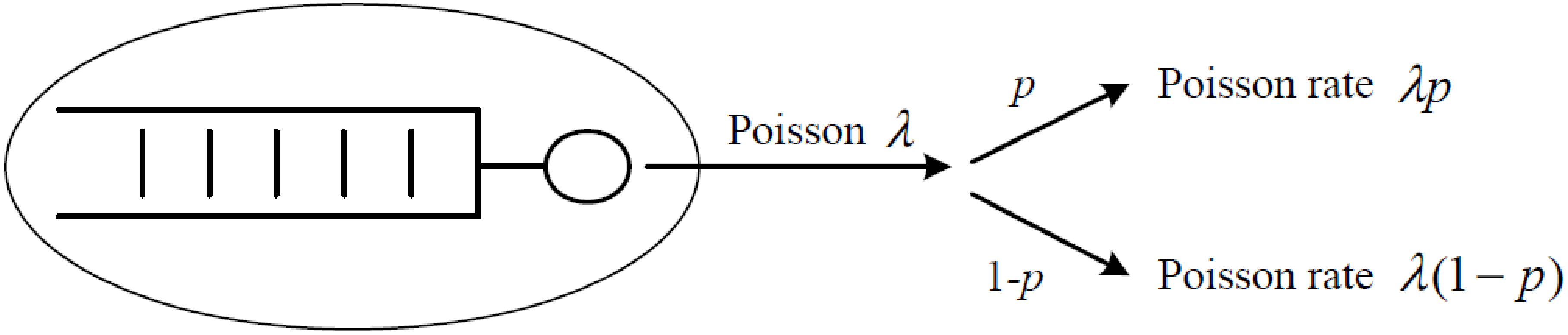


special case: If $p(s) = p$ is constant, then



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Proof. It is to prove that, for fixed time t,

$$\begin{aligned} P[\tilde{n}_1(t)=n,\tilde{n}_2(t)=m] &= P[\tilde{n}_1(t)=n]P[\tilde{n}_2(t)=m] \\ &= \frac{e^{-\lambda pt}(\lambda pt)^n}{n!}\cdot\frac{e^{-\lambda(1-p)t}[\lambda(1-p)t]^m}{m!} \end{aligned}$$

....

$$\begin{aligned} & P[\tilde{n}_1(t)=n,\tilde{n}_2(t)=m] \\ = & \sum_{k=0}^{\infty} P[\tilde{n}_1(t)=n,\tilde{n}_2(t)=m|\tilde{n}_1(t)+\tilde{n}_2(t)=k]\cdot P[\tilde{n}_1(t)+\tilde{n}_2(t)=k] \\ = & P[\tilde{n}_1(t)=n,\tilde{n}_2(t)=m|\tilde{n}_1(t)+\tilde{n}_2(t)=n+m]\cdot P[\tilde{n}_1(t)+\tilde{n}_2(t)=n+m] \end{aligned}$$

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- From the "condition distribution of the arrival times", any event occurs at some time that is uniformly distributed, and is independent of other events.
- Consider that only one arrival occurs in the interval [0, *t*]:

$$
\begin{aligned}
& P[\text{type - 1 arrival}|\tilde{n}(t)=1] \\
= \ & \int_0^t P[\text{type - 1 arrival}|\text{arrival time } \tilde{S}_1 = s, \tilde{n}(t)=1] \\
& \qquad \times \boxed{f_{\tilde{S}_1|\tilde{n}(t)}(s|\tilde{n}(t)=1)ds} \\
= \ & \int_0^t P(s)\cdot\frac{1}{t}ds = \frac{1}{t}\int_0^t P(s)ds = p
\end{aligned}
$$

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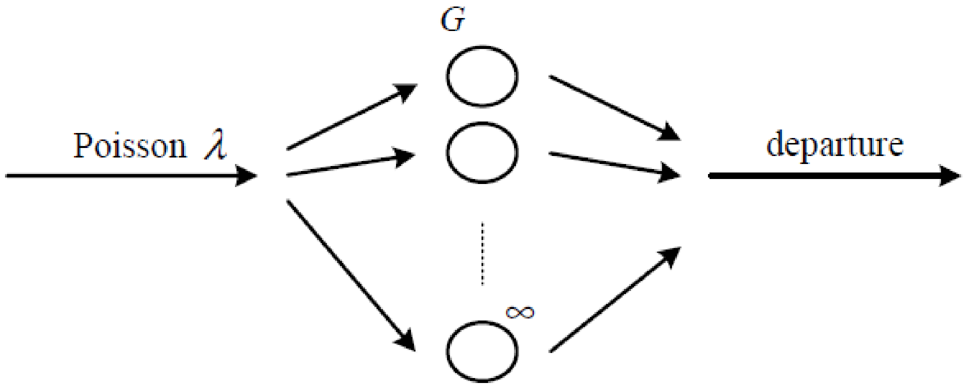
$$
\begin{aligned}
\therefore \quad & P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m] \\
= \quad & P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m | \tilde{n}_1(t) + \tilde{n}_2(t) = n + m] \cdot P[\tilde{n}_1(t) + \tilde{n}_2(t) = n + m] \\
= \quad & \begin{pmatrix} n + m \\ n \end{pmatrix} p^n (1-p)^m \cdot \frac{e^{-\lambda t} (\lambda t)^{n+m}}{(n+m)!} \\
= \quad & \frac{(n+m)!}{n!m!} p^n (1-p)^m \cdot \frac{e^{-\lambda t} (\lambda t)^{n+m}}{(n+m)!} \\
= \quad & \frac{e^{-\lambda p t} (\lambda p t)^n}{n!} \cdot \frac{e^{-\lambda (1-p) t} [\lambda (1-p) t]^m}{m!}
\end{aligned}
$$

Decomposition of a Poisson Process

- **Example** (An Infinite Server Queue, textbook [Ross]).



- $G_{\tilde{s}}(t) = P(\tilde{S} \le t)$, where $\tilde{S}$ = service time
- $G_{\tilde{s}}(t)$ is independent of each other and of the arrival process
- $\tilde{n}_1(t)$: the number of customers which have left before t;
- $\tilde{n}_2(t)$: the number of customers which are still in the system at time t;

$\Rightarrow \tilde{n}_1(t) \sim ?$ and $\tilde{n}_2(t) \sim ?$

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- **Answer.**
- $\widetilde{n_1}(t)$: the number of type-1 customers
- $\widetilde{n_2}(t)$: the number of type-2 customers

$$
\begin{aligned}
\text{type-1:} \quad P(s) &= P(\text{finish before } t) \\
&= P(\tilde{S} \le t - s) = G_{\tilde{s}}(t-s) \\
\text{type-2:} \quad 1 - P(s) &= \bar{G}_{\tilde{s}}(t-s)
\end{aligned}
$$

$$
\therefore \qquad \tilde{n}_1(t) \sim P\left(k; \lambda t \cdot \frac{1}{t}\int_0^t G_{\tilde{s}}(t-s)ds\right)
$$

$$
\tilde{n}_2(t) \sim P\left(k; \lambda t \cdot \frac{1}{t}\int_0^t \bar{G}_{\tilde{s}}(t-s)ds\right)
$$

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$$\therefore \quad E[\tilde{n}_1(t)] = \lambda t \cdot \frac{1}{t}\int_0^t G(t-s)ds$$

$$= \lambda \int_t^0 G(y)(-dy) \qquad \begin{aligned} t - s &= y \\ s &= t - y \end{aligned}$$

$$= \lambda \int_0^t G(y)dy$$

As $t \to \infty$, we have

$$\lim_{t\to\infty} E[\tilde{n}_2(t)] = \lambda \int_0^t \bar{G}(y)dy = \lambda E[\tilde{S}] \quad \text{(Little's formula)}$$