Chapter 2

Probability, Statistics, and Traffic Theories

Outline

- Introduction
- Probability Theory and Statistics Theory
  - Random variables
  - Probability mass function (pmf)
  - Probability density function (pdf)
  - Cumulative distribution function (cdf)
  - Expected value, n^{th} moment, n^{th} central moment, and variance
  - Some important distributions
- Traffic Theory
  - Poisson arrival model, etc.
- Basic Queuing Systems
  - Little's law
  - Basic queuing models

Introduction

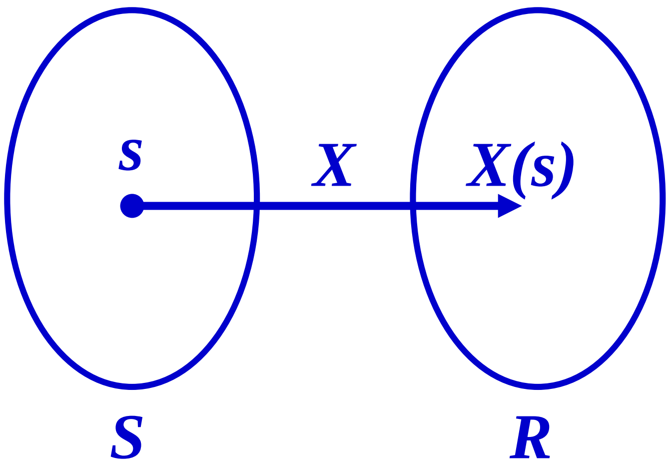
- Several factors influence the performance of wireless systems:
  - Density of mobile users
  - Cell size
  - Moving direction and speed of users (Mobility models)
  - Call rate, call duration
  - Interference, etc.
- Probability, statistics theory and traffic patterns, help make these factors tractable

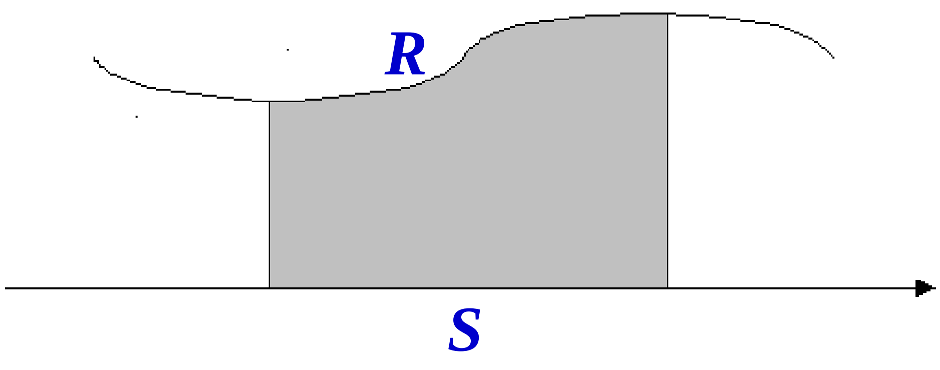
Probability Theory and Statistics Theory

- **Random Variables (RVs)**
  - **Let *S* be sample associated with experiment *E***
  - **<u>*X* is a function that associates a real number to each *s* ∈*S*</u>**
  - **RVs can be of two types: Discrete or Continuous**

Discrete random variable => probability mass function (pmf)

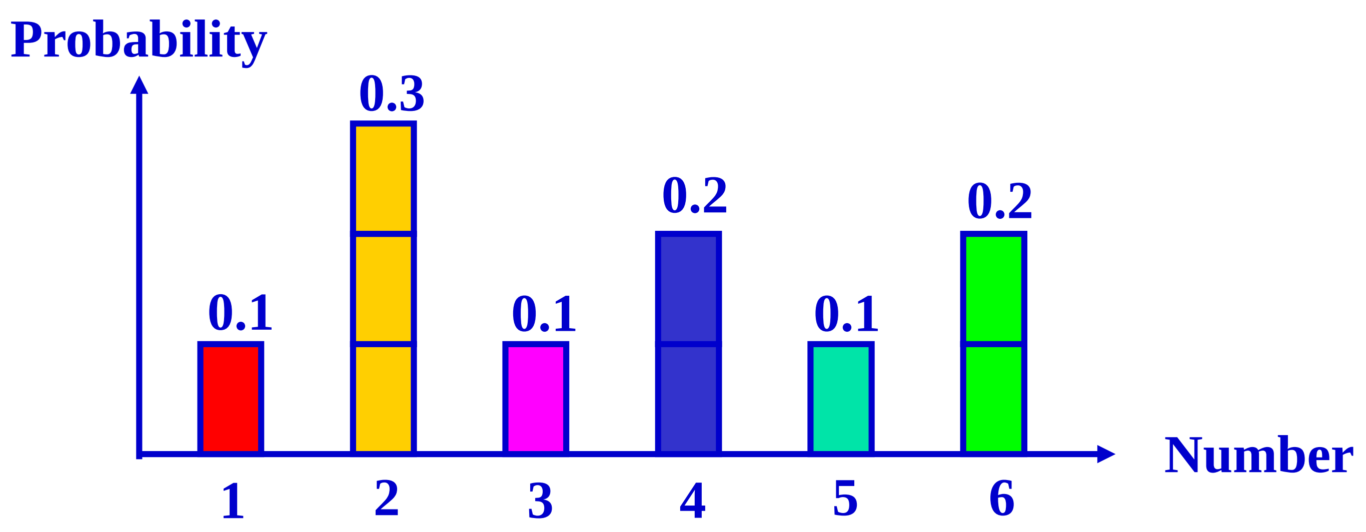
Continuous random variable => probability density function (pdf)





Discrete Random Variables

- In this case, *X(s)* contains a finite or infinite number of values
  - The possible values of *X* can be enumerated
- E.g., throw a 6 sided dice and calculate the probability of a particular number appearing.



Discrete Random Variables

- The probability mass function (pmf) $p(k)$ of X is defined as:

$$p(k) = p(X = k), \quad for\ k = 0, 1, 2, ...$$

where

1. Probability of each state occurring

$$0 \leq p(k) \leq 1, \text{ for every } k;$$

2. Sum of all states

$$\sum p(k) = 1, \text{ for all } k$$

Continuous Random Variables

- **In this case, *X* contains an infinite number of values**
- **Mathematically, *X* is a continuous random variable if there is a function *f*, called <u>probability density function (pdf)</u> of *X* that satisfies the following criteria:**

    1. $f(x) \geq 0$, **for all** x;

    2. $\int f(x)dx = 1$

Cumulative Distribution Function

- **Applies to all random variables**
- **A <u>cumulative distribution function (cdf)</u> is defined as:**
  - **For discrete random variables:**

$$P(k) = P(X \leq k) = \sum_{\text{all} \leq k} P(X = k)$$

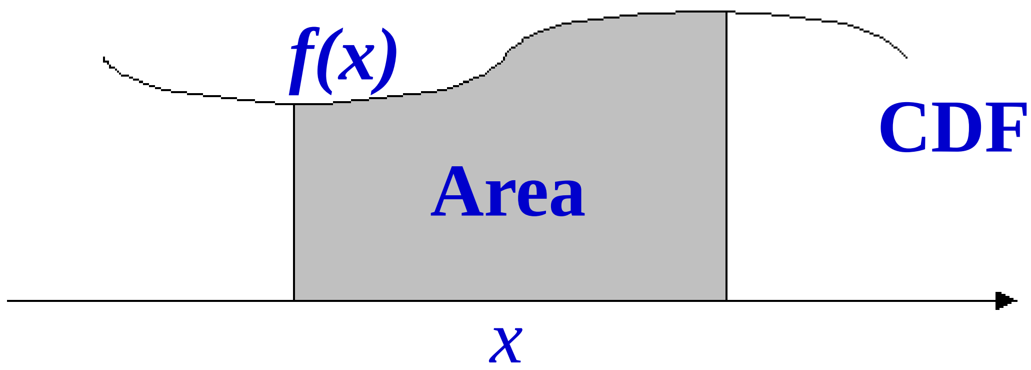
  - **For continuous random variables:**

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

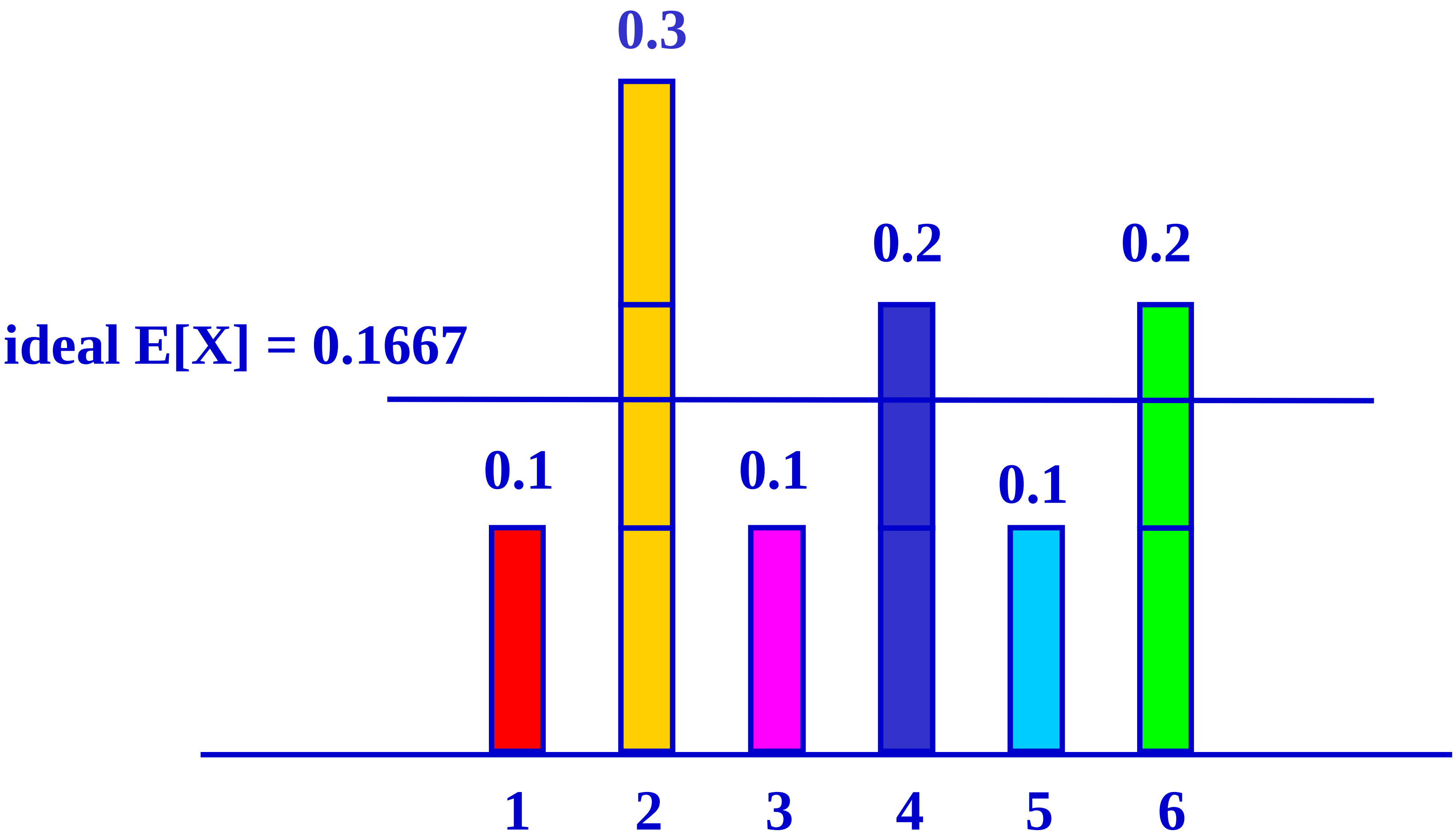
Probability Density Function

- **The pdf *f(x)* of a continuous random variable *X* is the derivative of the cdf *F(x)*, i.e.,**

$$f(x) = \frac{dF_X(x)}{dx}$$



Expected Value, n^{th} Moment, n^{th} Central Moment, and Variance



Expected Value, n^{th} Moment, n^{th} Central Moment, and Variance

- **Discrete Random Variables**
  - **Expected value represented by E or average of random variable**

$$E[X] = \sum_{\text{all} \le k} kP(X = k)$$

  - **n^{th} moment**

$$E[X^n] = \sum_{\text{all} \le k} k^n P(X = k)$$

  - **n^{th} central moment**

$$E[(X - E[X])^n] = \sum_{\text{all} \le k} (k - E[X])^n P(X = k)$$

  - **Variance or the second central moment**

$$\sigma^2 = Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Expected Value, n^{th} Moment, n^{th} Central Moment, and Variance

- **Continuous Random Variable**
  - **Expected value or mean value**

$$E[X] = \int_{-\infty}^{+\infty} x f(x) dx$$

  - **n^{th} moment**

$$E[X^n] = \int_{-\infty}^{+\infty} x^n f(x) dx$$

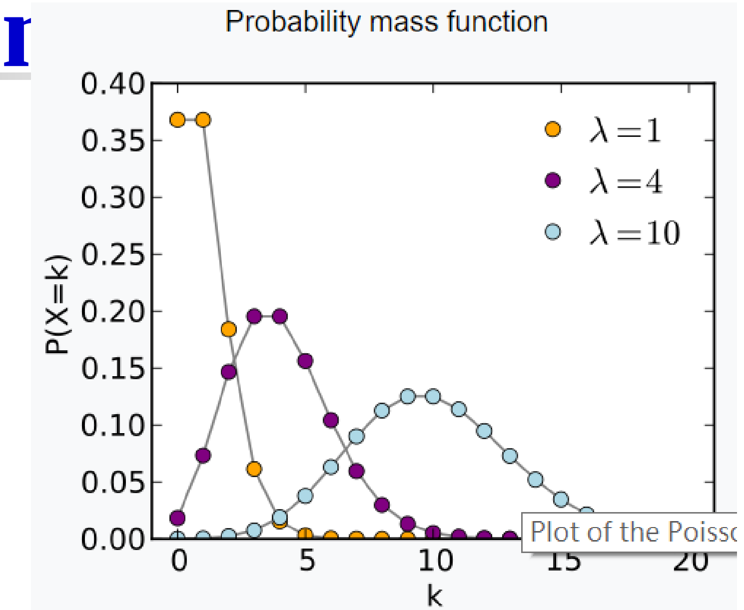
  - **n^{th} central moment**

$$E[(X - E[X])^n] = \int_{-\infty}^{+\infty} (x - E[X])^n f(x) dx$$

  - **Variance or the second central moment**

$$\sigma^2 = Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Some Important Discrete Random Distributions



- **Poisson**

$$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!},\ k=0,1,2,\ldots,\ and\ \lambda>0$$

  - ***E[X] = λ*, and *Var(X) = λ***

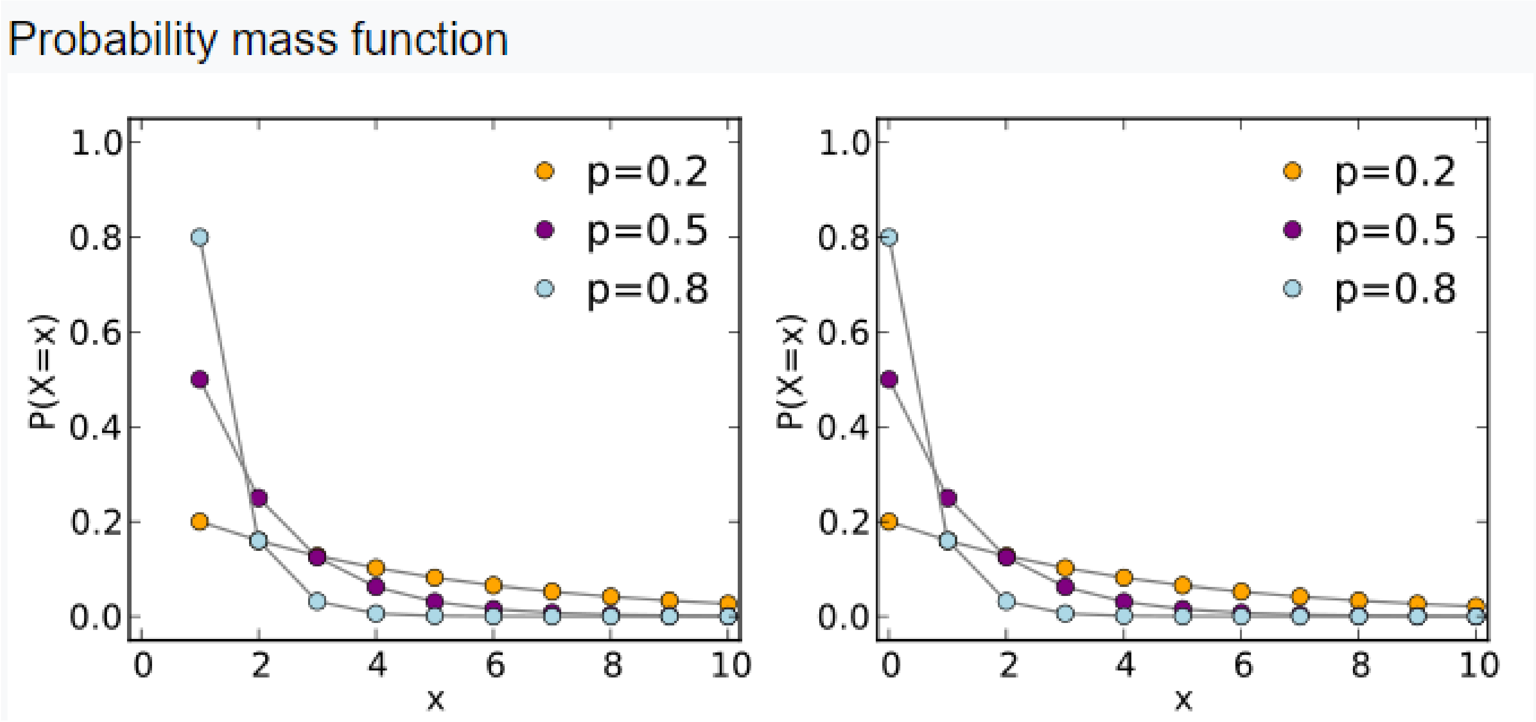
- **Geometric**

$$P(X=k) = p(1-p)^{k-1}\ ,$$

where p is success probability

  - ***E[X] = 1/(1-p)*, and *Var(X) = p/(1-p)²***

Geometric Distribution



Some Important Discrete Random Distributions

- **Binomial**

Out of *n* dice, exactly *k* dice have the same value: probability p^k and *(n-k)* dice have different values: probability $(1-p)^{n-k}$.

For any *k* dice out of *n*:

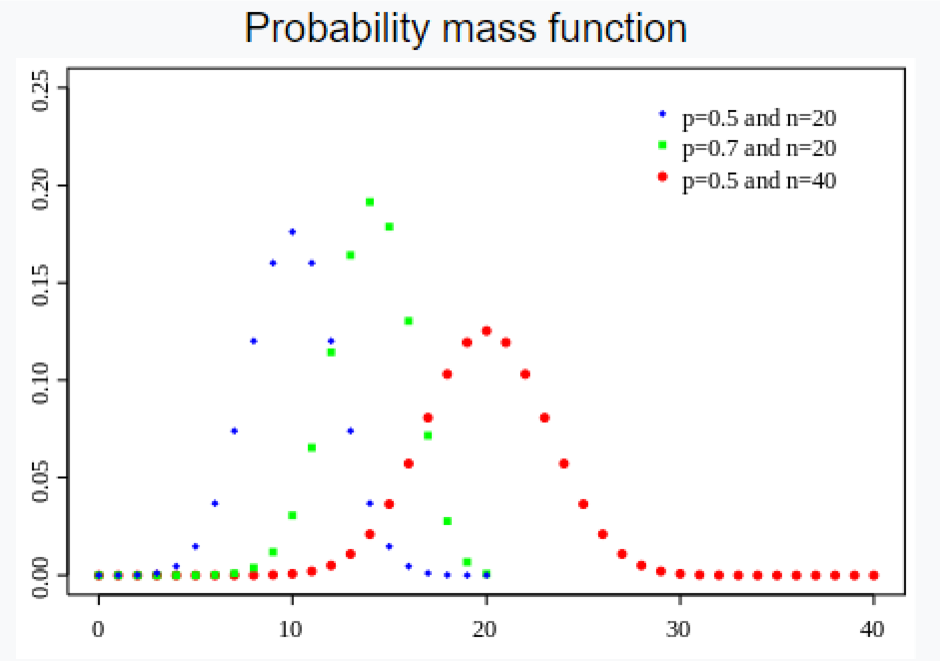
$$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k},$$

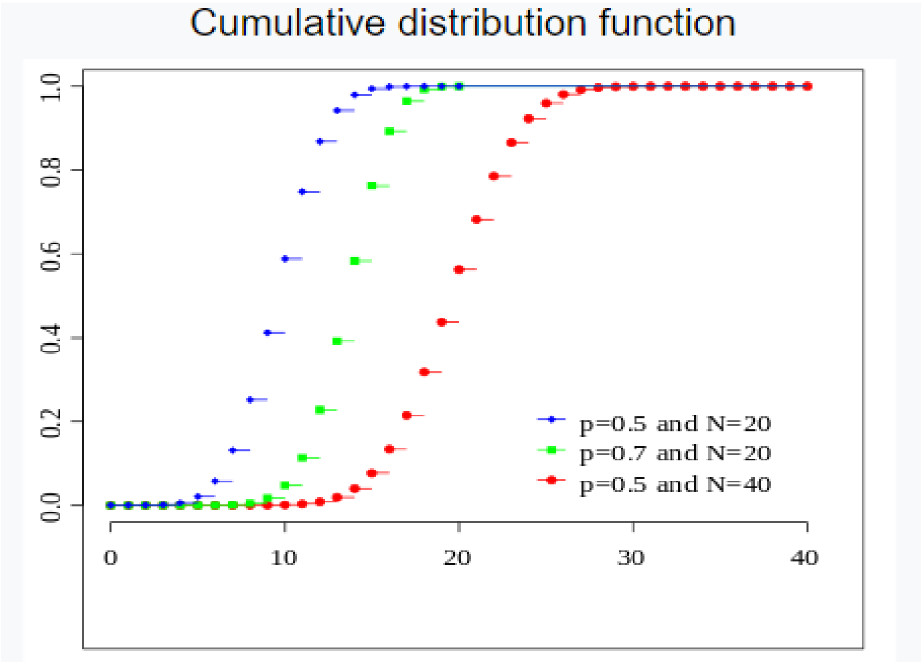
where,

$$k=0,1,2,\ldots,n;\ n=0,1,2,\ldots;\ p \text{ is the sucess probability, and}$$

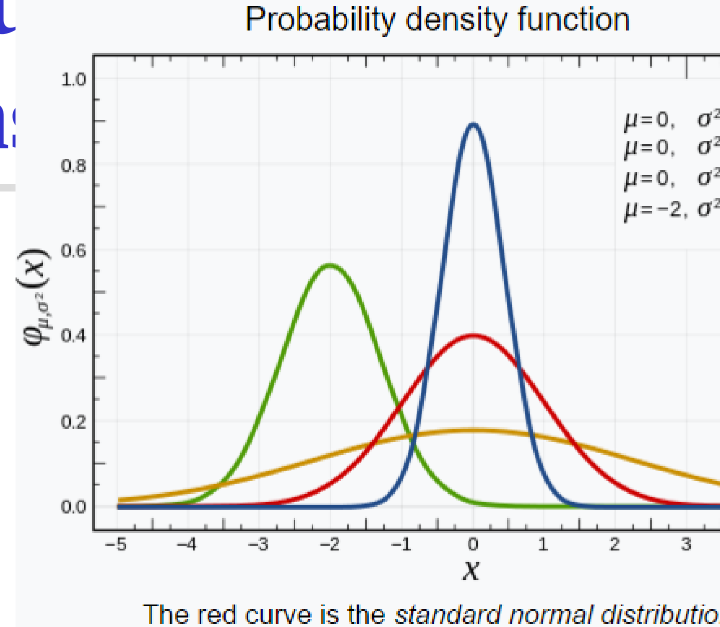
$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

Binomial Distribution





Some Important Continuous Random Distributions

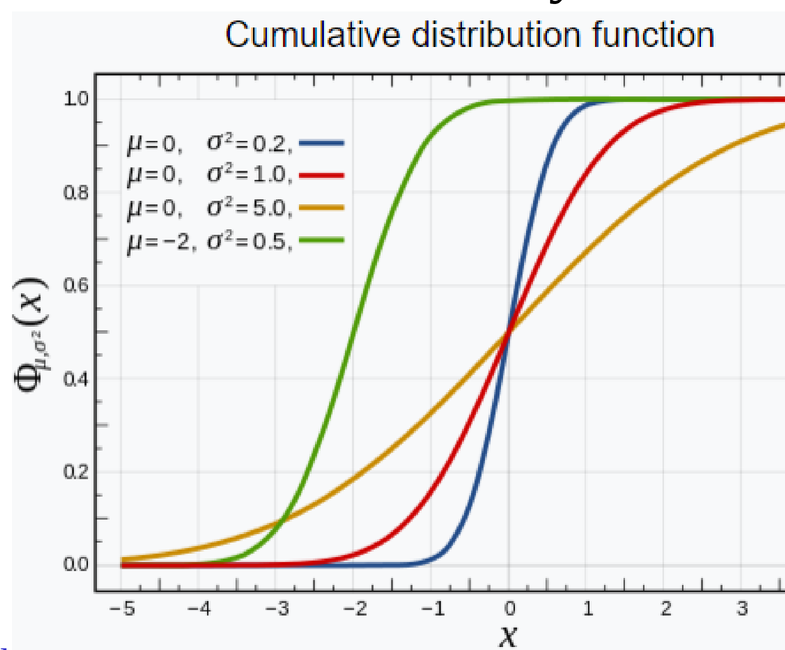


- **Normal**

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \text{ for } -\infty < x < \infty$$

and the cumulative distribution function can be obtained by

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy$$

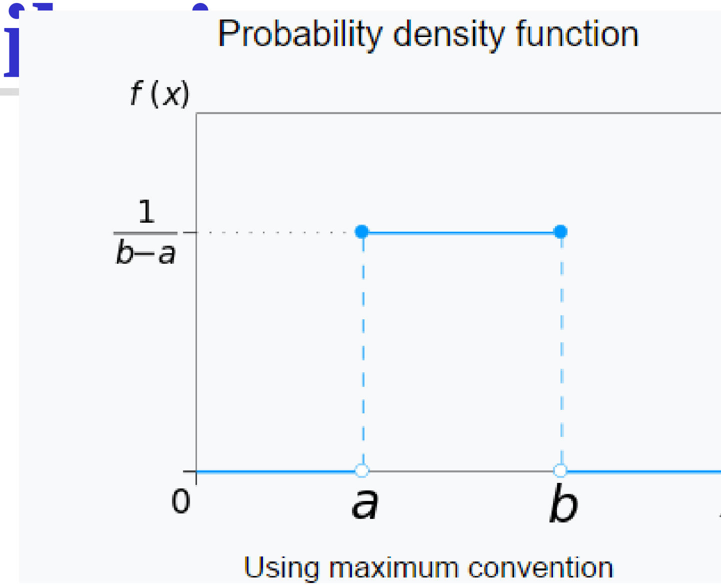


- ***E[X]* = *μ*, and *Var(X)* = *σ*²**

Some Important Continuous Random Distri

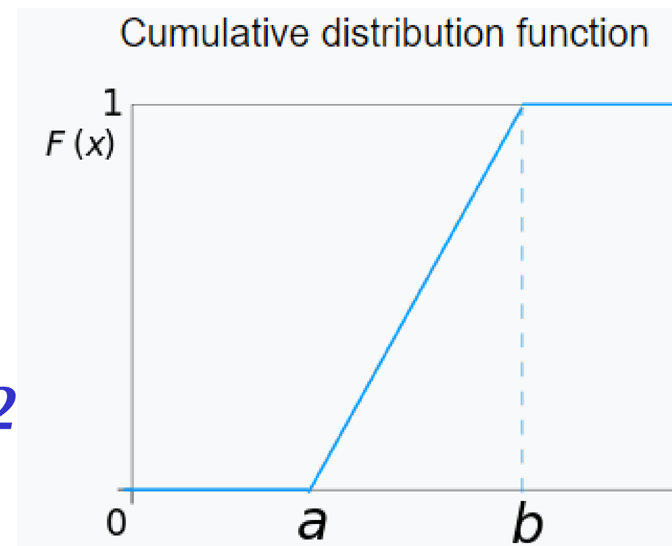
- **Uniform**

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & \text{for } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$



and the cumulative distribution function is

$$F_X(x) = \begin{cases} 0, & \text{for } x < a \\ \dfrac{x-a}{b-a}, & \text{for } a \le x \le b \\ 1, & \text{for x > b} \end{cases}$$

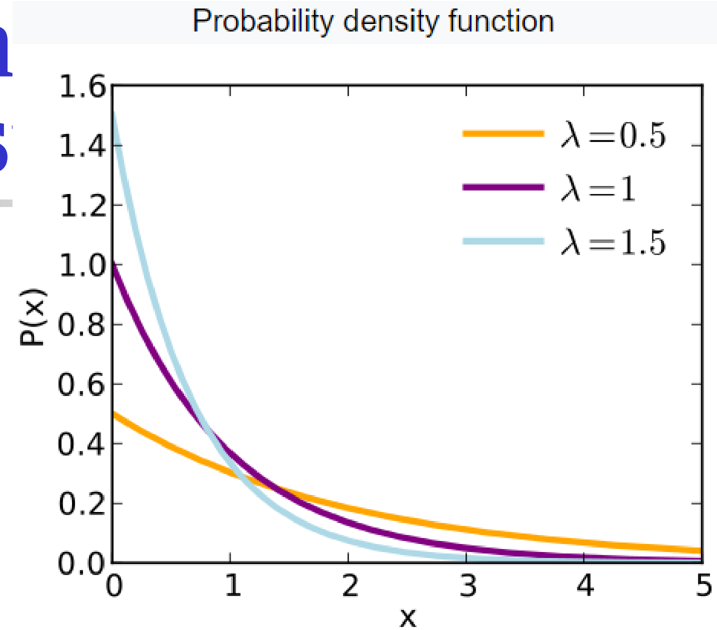


- ***E[X] = (a+b)/2*, and *Var(X) = (b-a)^2/12***

Some Important Random Dis

- **Exponential**

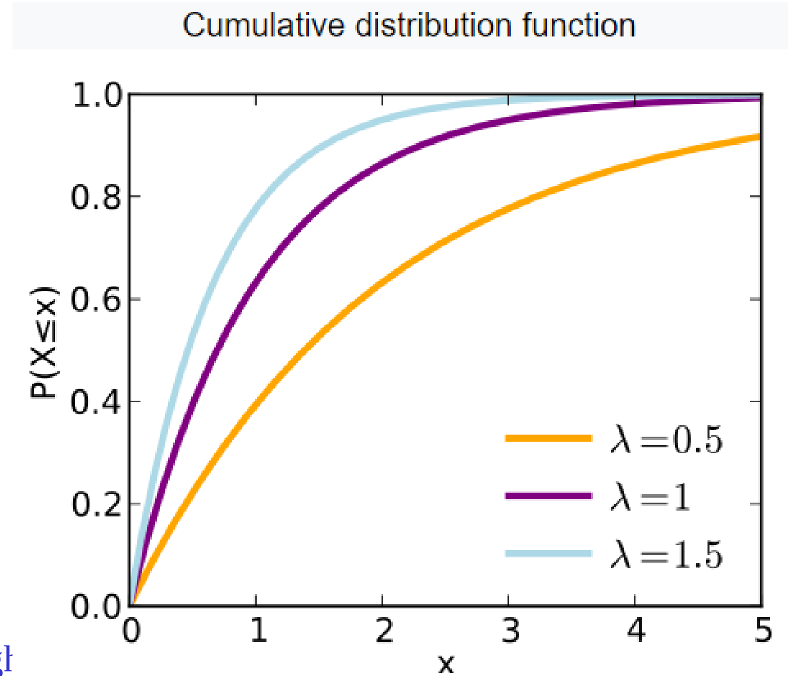
$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & \text{for } 0 \le x < \infty \end{cases}$$



and the cumulative distribution function is

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & \text{for } 0 \le x < \infty \end{cases}$$

- ***E[X] = 1/λ*, and *Var(X) = 1/λ²***



Multiple Random Variables

- **There are cases where the result of one experiment determines the values of several random variables**
- **The joint probabilities of these variables are:**
  - **Discrete variables:**

    $$p(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$$

  - **Continuous variables:**

    cdf: $$F_{x1x2\ldots xn}(x_1, \ldots, x_n) = P(X_1 \le x_1, \ldots, X_n \le x_n)$$

    pdf: $$f_{X_1, X_2, \ldots X_n}(x_1, x_2, \ldots x_n) = \frac{\partial^n F_{X_1, X_2, \ldots X_n}(x_1, x_2, \ldots x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

Independence and Conditional Probability

- **Independence: The random variables are said to be independent of each other when the occurrence of one does not affect the other.**
- **The pmf for discrete random variables in such a case is given by: $p(x_1,x_2,\ldots x_n)=P(X_1=x_1)P(X_2=x_2)\ldots P(X_3=x_3)$ and for continuous random variables as:**

  $F_{X1,X2,\ldots Xn} = F_{X1}(x_1)F_{X2}(x_2)\ldots F_{Xn}(x_n)$
- **Conditional probability: is the probability that $X_1= x_1$ given that $X_2= x_2$.**
- **Then for discrete random variables the probability becomes:**

$$P(X_1 = x_1 \mid X_2 = x_2, \ldots, X_n = x_n) = \frac{P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n)}{P(X_2 = x_2, \ldots, X_n = x_n)}$$

and for continuous random variables it is:

$$P(X_1 \le x_1 \mid X_2 \le x_2, \ldots, X_n \le x_n) = \frac{P(X_1 \le x_1, X_2 \le x_2, \ldots, X_n \le x_n)}{P(X_2 \le x_2, \ldots X_n \le x_n)}$$

Bayes Theorem

- **A theorem concerning conditional probabilities of the form *P(X|Y)* (read: the probability of *X*, given *Y*) is**

$$P(X \mid Y) = \frac{P(Y \mid X)P(X)}{P(Y)}$$

where *P(X)* and *P(Y)* are the unconditional probabilities of *X* and *Y* respectively

Important Properties of Random Variables

- **Sum property of the expected value**
  - **Expected value of the sum of random variables:**

$$E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i]$$

- **Product property of the expected value**
  - **Expected value of product of stochastically independent random variables**

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i]$$

Important Properties of Random Variables

- **Sum property of the variance**
  - **Variance of the sum of random variables is**

$$Var\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i a_j \operatorname{cov}[X_i, X_j]$$

where $cov[X_i, X_j]$ is the covariance of random variables X_i and X_j and

$$\operatorname{cov}[X_i, X_j] = E[(X_i - E[X_i])(X_j - E[X_j])] = E[X_i X_j] - E[X_i]E[X_j]$$

If random variables are independent of each other, i.e., $cov[X_i, X_j]=0$, then

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

Important Properties of Random Variables

- **Distribution of sum - For continuous random variables with joint pdf $f_{XY}(x, y)$ and if $Z = \Phi(X,Y)$, the distribution of Z may be written as**

$$F_Z(z) = P(Z \le z) = \int_{\phi Z} f_{XY}(x, y)dxdy$$

where Φ_Z *is a subset of Z.*

- **For a special case $Z = X+Y$**

$$F_Z(z) = \iint_{\phi Z} f_{XY}(x, y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)dxdy$$

- **If X and Y are independent variables, the $f_{XY}(x,y) = f_X(x)f_Y(y)$**

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)dx, \quad \text{for } -\infty \le z < \infty$$

- **If both X and Y are non negative random variables, then pdf is the convolution of the individual pdfs, $f_X(x)$ and $f_Y(y)$**

$$f_Z(z) = \int_0^z f_X(x) f_Y(z-x)dx, \quad \text{for } -\infty \le z < \infty$$

Central Limit Theorem

The ***Central Limit Theorem*** **states that whenever a random sample (X_1, X_2,.. X_n) of size n is taken from any distribution with expected value $E[X_i] = \mu$ and variance $Var(X_i) = \sigma^2$, where i =1,2,..,n, then their arithmetic mean is defined by**

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Central Limit Theorem

- Let $\{X_1, \ldots, X_n\}$ be a random sample of size n — that is, a sequence of independent and identically distributed random variables drawn from distributions of expected values given by μ and finite variances given by σ^2.
- Suppose we are interested in the sample average of these random variables.

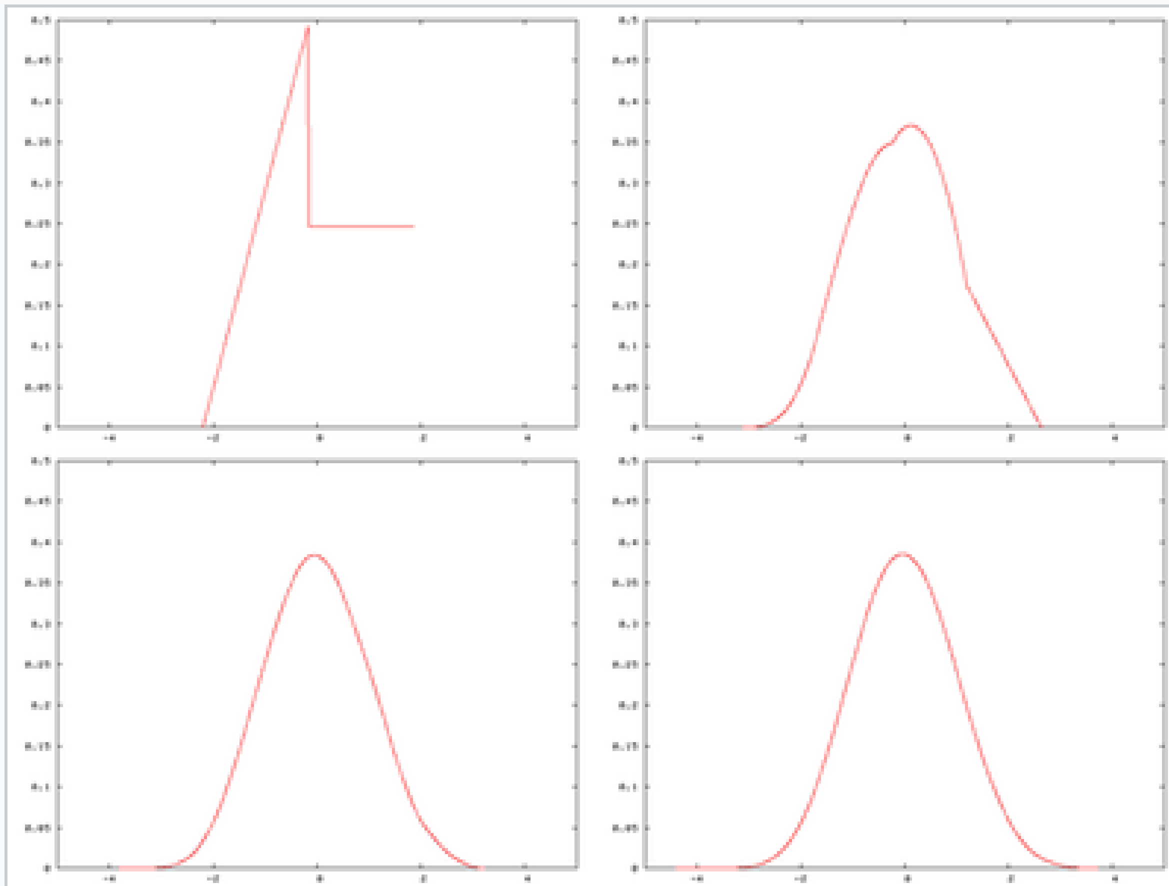
$$S_n := \frac{X_1 + \cdots + X_n}{n}$$

- By the law of large numbers, the sample averages converge in probability and almost surely to the expected value μ as $n \to \infty$

Central Limit Theorem

- **The sample mean is approximated to a normal distribution with**
  - $E[S_n] = \mu$, **and**
  - $Var(S_n) = \sigma^2 / n$
- **The larger the value of the sample size *n*, the better the approximation to the normal**
- **This is very useful when inference between signals needs to be considered**

Central Limit Theorem



A distribution being "smoothed out" by summation, showing original density of distribution and three subsequent summations; see Illustration of the central limit theorem for further details.