

Chapter 4 Poisson Processes - 2

Non-homogeneous Poisson Processes

- The counting process $N = \{\tilde{n}(t), t \geq 0\}$ is said to be a *non-stationary* or *non-homogeneous* Poisson Process with time-varying intensity 與時間相關 function $\lambda(t), t \geq 0$, if:

1. $\tilde{n}(0) = 0$

2. N has independent increments

3. $P[\tilde{n}(t+h) - \tilde{n}(t) \geq 2] = o(h)$

4. $P[\tilde{n}(t+h) - \tilde{n}(t) = 1] = \lambda(t) \cdot h + o(h)$

Proof. < Homework >.

$$\begin{aligned} P(X(h)=1) &= e^{-\lambda h} (\lambda h) \\ &= \lambda h(1 - \lambda h + o(h)) \\ &= \lambda h + o(h) \end{aligned}$$

- Define “integrated intensity function” $m(t) = \int_0^t \lambda(t') dt'$.

Theorem.

$$P[\tilde{n}(t+s) - \tilde{n}(t) = n] = \frac{e^{-[m(t+s)-m(t)]} [m(t+s) - m(t)]^n}{n!}$$

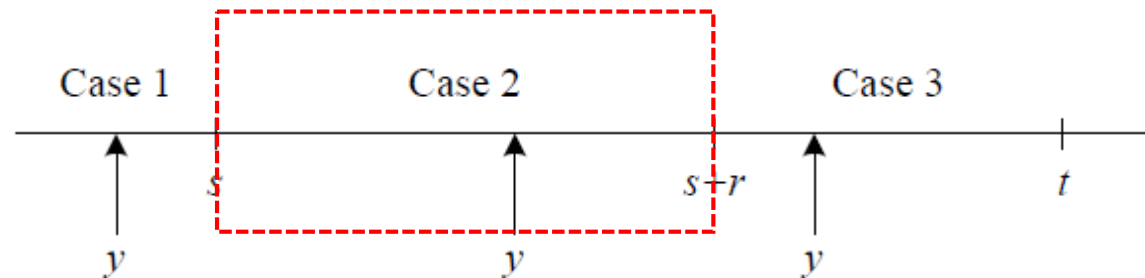
Non-homogeneous Poisson Processes

- **Example.** The “output process” of the $M/G/\infty$ queue is a non-homogeneous Poisson process having intensity function $\lambda(t) = \lambda G(t)$, where G is the service distribution.
- **Hint.** Let $D(s, s + r)$ denote the number of service completions in the interval $(s, s + r]$ in $(0, t]$. If we can show that
 - $D(s, s + r)$ follows a Poisson distribution with mean $\lambda \int_s^{s+r} G(y)dy$ and
 - the numbers of service completions in disjoint intervals are independent,
- Then we are finished by definition of a non-homogeneous Poisson process

Non-homogeneous Poisson Processes

- **Answer.**

- An arrival at time y is called a type-1 arrival if its service completion occurs in $(s, s + r]$.
- Consider three cases to find the probability $P(y)$ that an arrival at time y is a type-1 arrival:



– Case 1: $y \leq s$.

$$P(y) = P\{s - y < \tilde{S} < s + r - y\} = G(s + r - y) - G(s - y)$$

– Case 2: $s < y \leq s + r$.

$$P(y) = P\{\tilde{S} < s + r - y\} = G(s + r - y)$$

Non-homogeneous Poisson Processes

– Case 3: $s + r < y \leq t$.

$$P(y) = 0$$

- Based on the decomposition property of a Poisson process, we conclude that $D(s, s + r)$ follows a Poisson distribution with mean $\lambda p t$, where $p = (1/t) \int_0^t P(y) dy$

$$\begin{aligned} \int_0^t P(y) dy &= \int_0^s [G(s + r - y) - G(s - y)] dy + \int_s^{s+r} G(s + r - y) dy \\ &\quad + \int_{s+r}^t (0) dy \\ &= \int_0^{s+r} G(s + r - y) dy - \int_0^s G(s - y) dy \\ &= \int_0^{s+r} G(z) dz - \int_0^s G(z) dz = \int_s^{s+r} G(z) dz \end{aligned}$$

Non-homogeneous Poisson Processes

- Because of
 - the independent increment assumption of the Poisson arrival process, and
 - the fact that there are always servers available for arrivals,

⇒ the departure process has independent increments

Compound Poisson Processes

- A stochastic process $\{\tilde{x}(t), t \geq 0\}$ is said to be a *compound Poisson process* if

- it can be represented as

$$\tilde{x}(t) = \sum_{i=1}^{\tilde{n}(t)} \tilde{y}_i, \quad t \geq 0$$

- $\{\tilde{n}(t), t \geq 0\}$ is a Poisson process
- $\{\tilde{y}_i, i \geq 1\}$ is a family of independent and identically distributed random variables which are also independent of $\{\tilde{n}(t), t \geq 0\}$
- The random variable $\tilde{x}(t)$ is said to be a compound Poisson random variable.
- $E[\tilde{x}(t)] = \lambda t E[\tilde{y}_i]$ and $Var[\tilde{x}(t)] = \lambda t E[\tilde{y}_i^2]$.

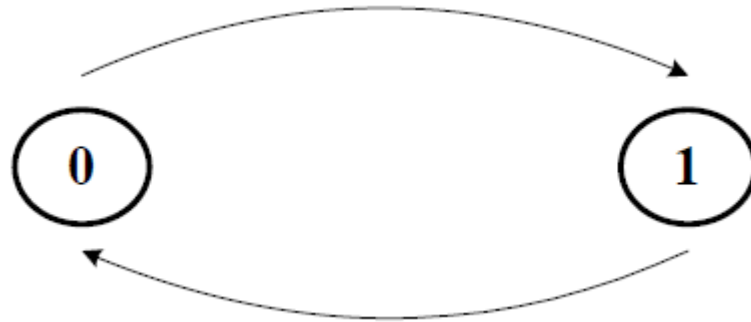
Compound Poisson Processes

- **Example** (Batch Arrival Process). Consider a parallel-processing system where each job arrival consists of a possibly random number of tasks. Then we can model the arrival process as a *compound Poisson process*, which is also called a *batch arrival process*.
- Let \tilde{y}_i be a random variable that denotes *the number of tasks comprising a job*. We derive the probability generating function $P_{\tilde{x}(t)}(z)$ as follows:

$$\begin{aligned} P_{\tilde{x}(t)}(z) &= E \left[z^{\tilde{x}(t)} \right] = E \left[E \left[z^{\tilde{x}(t)} \mid \tilde{n}(t) \right] \right] = E \left[E \left[z^{\tilde{y}_1 + \dots + \tilde{y}_{\tilde{n}(t)}} \mid \tilde{n}(t) \right] \right] \\ &= E \left[E \left[z^{\tilde{y}_1 + \dots + \tilde{y}_{\tilde{n}(t)}} \right] \right] \quad (\text{by independence of } \tilde{n}(t) \text{ and } \{\tilde{y}_i\}) \\ &= E \left[E \left[z^{\tilde{y}_1} \right] \dots E \left[z^{\tilde{y}_{\tilde{n}(t)}} \right] \right] \quad (\text{by independence of } \tilde{y}_1, \dots, \tilde{y}_{\tilde{n}(t)}) \\ &= E \left[(P_{\tilde{y}}(z))^{\tilde{n}(t)} \right] = P_{\tilde{n}(t)}(P_{\tilde{y}}(z)) \end{aligned}$$

Modulated Poisson Processes

- Assume that there are two states, 0 and 1, for a “modulating process.”



- When the state of the modulating process equals 0 then the arrive rate of customers is given by λ_0 , and when it equals 1 then the arrival rate is λ_1 .
- The residence time in a particular modulating state is exponentially distributed with parameter μ and, after expiration of this time, the modulating process changes state.
- The initial state of the modulating process is randomly selected and is equally likely to be state 0 or 1.

Modulated Poisson Processes

待在此state 0 的時間

- For a given period of time $(0, t)$, let Y be a random variable that indicates the total amount of time that the modulating process has been in state 0. Let $\tilde{x}(t)$ be the number of arrivals in $(0, t)$.
- Then, given Y , the value of $\tilde{x}(t)$ is distributed as a non-homogeneous Poisson process and thus

$$P[\tilde{x}(t) = n | Y = \tau] = \frac{(\lambda_0 \tau + \lambda_1 (t - \tau))^n e^{-(\lambda_0 \tau + \lambda_1 (t - \tau))}}{n!}$$

- As $\mu \rightarrow 0$, the probability that the modulating process makes no transitions within t seconds converges to 1, and we expect for this case that

$$P[\tilde{x}(t) = n] = \frac{1}{2} \left\{ \frac{(\lambda_0 t)^n e^{-\lambda_0 t}}{n!} + \frac{(\lambda_1 t)^n e^{-\lambda_1 t}}{n!} \right\}$$

Modulated Poisson Processes

- As $\mu \rightarrow \infty$, then the modulating process makes an infinite number of transitions within t seconds, and we expect for this case that

$$P[\tilde{x}(t) = n] = \frac{(\beta t)^n e^{-\beta t}}{n!}, \quad \text{where } \beta = \frac{\lambda_0 + \lambda_1}{2}$$

- **Example** (Modeling Voice).
 - A basic feature of speech is that it comprises an alternation of silent periods and non-silent periods.
 - The arrival rate of packets during a talk spurt period is Poisson with rate λ_1 and silent periods produce a Poisson rate with $\lambda_0 \approx 0$.
 - The duration of time for talk and silent periods are exponentially distributed with parameters μ_1 and μ_0 , respectively.
- ⇒ The model of the arrival stream of packets is given by a modulated Poisson process.

Poisson Arrivals See Time Averages (PASTA)

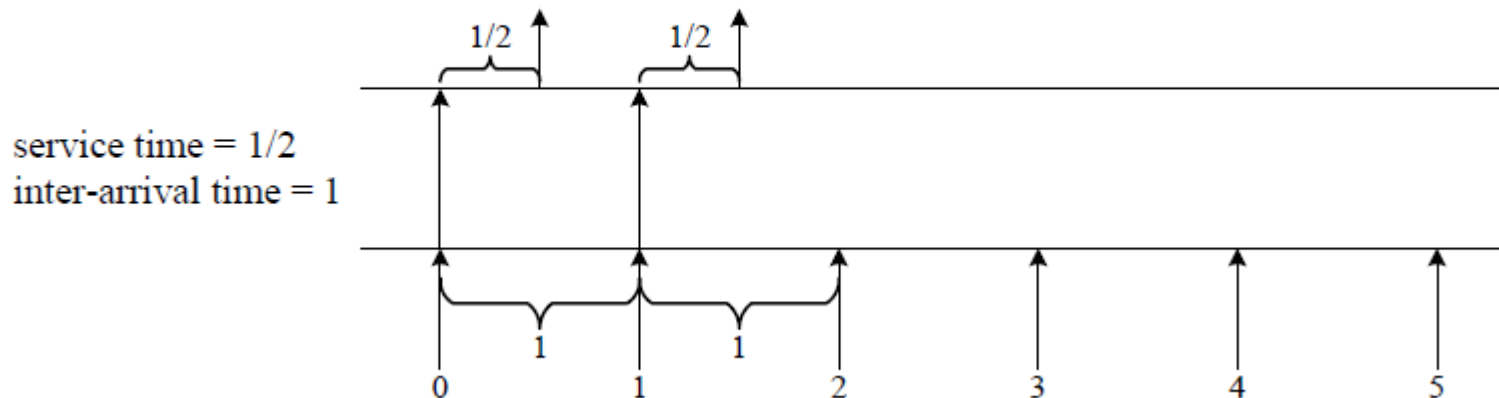
- PASTA says: as $t \rightarrow \infty$

Fraction of arrivals who see the system in a given state upon arrival (arrival average)

= Fraction of time for the system is in a given state (time average)

= The system is in the given state at any random time after being steady

Counter-example (textbook [Kao]: Example 2.7.1)



Poisson Arrivals See Time Averages (PASTA)

- Arrival average that an arrival will see an idle system = 1
- Time average of system being idle = $\frac{1}{2}$

Mathematically,

- Let $X = \{\tilde{x}(t), t \geq 0\}$ be a stochastic process with state space S , and $B \subset S$
- Define an indicator random variable
$$\tilde{u}(t) = \begin{cases} 1, & \text{if } \tilde{x}(t) \in B \\ 0, & \text{otherwise} \end{cases}$$
- Let $N = \{\tilde{n}(t), t \geq 0\}$ be a Poisson process with rate λ denoting the arrival process

Poisson Arrivals See Time Averages (PASTA)

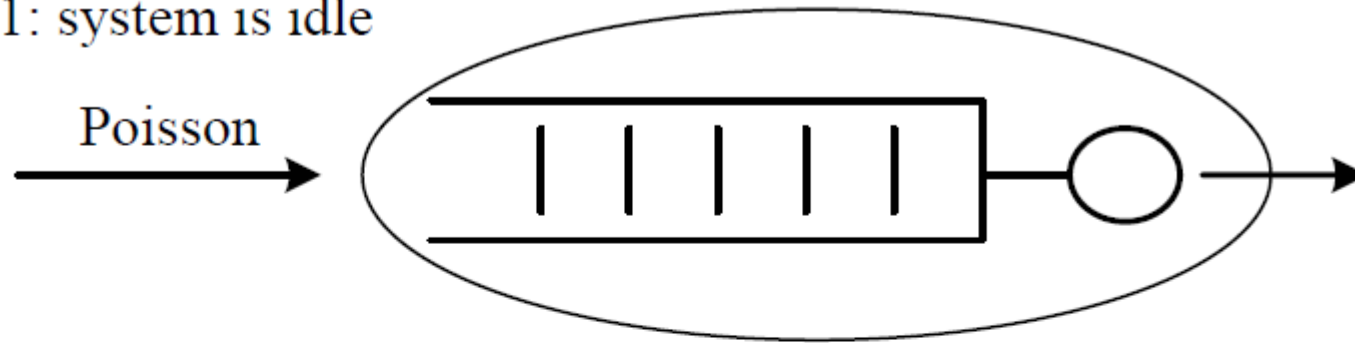
$$\lim_{t \rightarrow \infty} \frac{\int_0^t \tilde{u}(s) d\tilde{n}(s)}{\tilde{n}(t)} = \lim_{t \rightarrow \infty} \frac{\int_0^t \tilde{u}(s) ds}{t}$$

(arrival average) *(time average)*

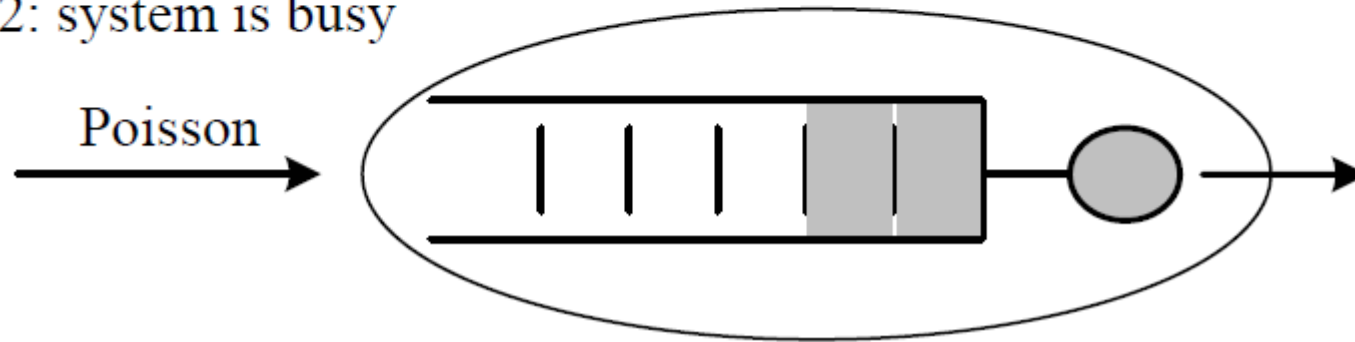
- Condition – For PASTA to hold, we need the *lack of anticipation assumption* (LAA): for each $t \geq 0$,
 - the arrival process $\{\tilde{n}(t+u) - \tilde{n}(t), u \geq 0\}$ is independent of $\{\tilde{x}(s), 0 \leq s \leq t\}$ and $\{\tilde{n}(s), 0 \leq s \leq t\}$.
- Application:
 - To find the waiting time distribution of any arriving customer
 - Given: $P[\text{system is idle}] = 1 - \rho$; $P[\text{system is busy}] = \rho$

Poisson Arrivals See Time Averages (PASTA)

Case 1: system is idle



Case 2: system is busy



$$\begin{aligned} \Rightarrow P(\tilde{w} \leq t) &= P(\tilde{w} \leq t | \text{idle}) \cdot P(\text{idle upon arrival}) \\ &+ P(\tilde{w} \leq t | \text{busy}) \cdot P(\text{busy upon arrival}) \end{aligned}$$

Memoryless Property of the Exponential Distribution

- A random variable \tilde{x} is said to be without memory, or *memoryless*, if

$$P[\tilde{x} > s + t | \tilde{x} > t] = P[\tilde{x} > s] \quad \text{for all } s, t \geq 0 \quad (3)$$

- The condition in Equation (3) is equivalent to

$$\frac{P[\tilde{x} > s + t, \tilde{x} > t]}{P[\tilde{x} > t]} = P[\tilde{x} > s]$$

or

$$P[\tilde{x} > s + t] = P[\tilde{x} > s]P[\tilde{x} > t] \quad (4)$$

- Since Equation (4) is satisfied when \tilde{x} is exponentially distributed (for $e^{-\lambda(s+t)} = e^{-\lambda s}e^{-\lambda t}$), it follows that exponential random variables are memoryless.
- Not only is the exponential distribution “memoryless,” but it is the unique continuous distribution possessing this property.

Exponential Distribution

- T_2 is the time between first and second arrivals, we define T_3 as the time between the second and third arrivals, T_4 as the time between the third and fourth arrivals and so on
- The random variables T_1, T_2, T_3, \dots are called the interarrival times of the Poisson process
- T_1, T_2, T_3, \dots are *independent* of each other and each has the same exponential distribution with mean arrival rate λ

Memoryless and Merging Properties

- Memoryless property
 - A random variable X has the property that “the future is independent of the past” i.e., the fact that it hasn't happened yet, tells us nothing about how much longer it will take before it does happen
- Merging property
 - If we merge n Poisson processes with distributions for the inter arrival times

$$1 - e^{-\lambda t} \text{ where } i = 1, 2, \dots, n$$

into one single process, then the result is a Poisson process for which the inter arrival times have the distribution $1 - e^{-\lambda t}$ with mean

$$\lambda = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

Memoryless

■ **Proof :**

$$\begin{aligned} P\{T > t + t_0 \mid T > t_0\} &= \frac{P\{T > t + t_0, T > t_0\}}{P\{T > t_0\}} = \frac{P\{T > t + t_0\}}{P\{T > t_0\}} \\ &= \frac{e^{-\alpha(t+t_0)}}{e^{-\alpha t_0}} = e^{-\alpha t} \\ &= P\{T > t\} \end{aligned}$$

Comparison of Two Exponential Random Variables

- Suppose that \tilde{x}_1 and \tilde{x}_2 are independent exponential random variables with respective means $1/\lambda_1$ and $1/\lambda_2$. What is $P[\tilde{x}_1 < \tilde{x}_2]$?

$$\begin{aligned} P[\tilde{x}_1 < \tilde{x}_2] &= \int_0^{\infty} P[\tilde{x}_1 < \tilde{x}_2 | \tilde{x}_1 = x] \lambda_1 e^{-\lambda_1 x} dx \\ &= \int_0^{\infty} P[x < \tilde{x}_2] \lambda_1 e^{-\lambda_1 x} dx \\ &= \int_0^{\infty} e^{-\lambda_2 x} \lambda_1 e^{-\lambda_1 x} dx \\ &= \int_0^{\infty} \lambda_1 e^{-(\lambda_1 + \lambda_2)x} dx \\ &= \frac{\lambda_1}{\lambda_1 + \lambda_2} \end{aligned}$$

Minimum of Exponential Random Variables

- Suppose that $\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n$ are independent exponential random variables, with \tilde{x}_i having rate μ_i , $i = 1, \dots, n$. It turns out that the smallest of the \tilde{x}_i is exponential with **a rate equal to the sum of the μ_i** .

$$\begin{aligned} P[\min(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n) > x] &= P[\tilde{x}_i > x \text{ for each } i = 1, \dots, n] \\ &= \prod_{i=1}^n P[\tilde{x}_i > x] \quad (\text{by independence}) \\ &= \prod_{i=1}^n e^{-\mu_i x} \\ &= \exp \left\{ - \left(\sum_{i=1}^n \mu_i \right) x \right\} \end{aligned}$$

How about $\max(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)$? (exercise)

Little's Law

- Assuming a queuing environment to be operating in a stable steady state where all initial transients have vanished, the key parameters characterizing the system are:
 - λ – the mean steady state consumer arrival
 - N – the average no. of customers in the system
 - T – the mean time spent by each customer in the system

which gives

$$N = \lambda T$$