Chapter 3 Poisson Processes

Outline

- Introduction to Poisson Processes
- Properties of Poisson processes
  - Inter-arrival time distribution
  - Waiting time distribution
  - Superposition and decomposition

兩個Poisson processes 相加

- Non-homogeneous Poisson processes (relaxing *stationary*)
- Compound Poisson processes (relaxing *single arrival*)
- Modulated Poisson processes (relaxing *independent*)
- Poisson Arrival See Time Average (PASTA)

Introduction

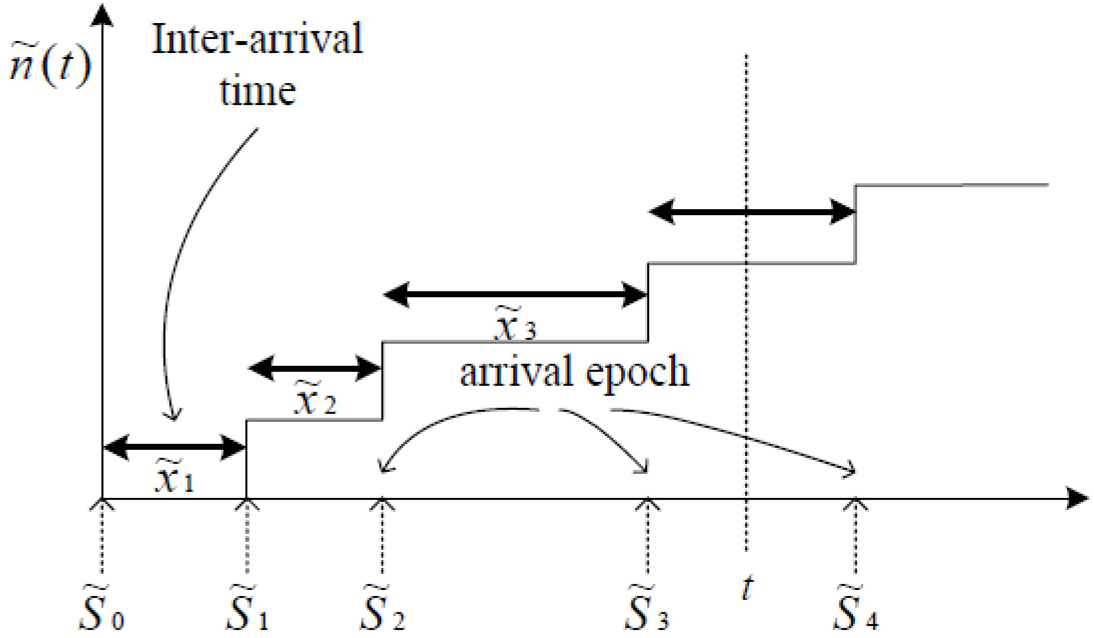
(i) n_{th} arrival epoch $\tilde{S}_n$ is

$$\tilde{S}_n = \tilde{x}_1 + \tilde{x}_2 + \ldots + \tilde{x}_n = \sum_{i=1}^{n} \tilde{x}_i$$

$$\tilde{S}_0 = 0$$

(ii) Number of arrivals at time t is: $\tilde{n}(t)$. Notice that:

$$\{\tilde{n}(t) \geq n\} \overset{iff}{\Leftrightarrow} \{\tilde{S}_n \leq t\},\ \{\tilde{n}(t) = n\} \overset{iff}{\Leftrightarrow} \{\tilde{S}_n \leq t \text{ and } \tilde{S}_{n+1} > t\}$$



Introduction

Arrival Process: $X = \{\tilde{x}_i,\ i = 1, 2, \ldots\};\ \tilde{x}_i$'s can be any

$S = \{\tilde{S}_i,\ i = 0, 1, 2, \ldots\}; \tilde{S}_i$'s can be any

$N = \{\tilde{n}(t),\ t \geq 0\}; \longrightarrow$ called arrival process

Renewal Process: $X = \{\tilde{x}_i,\ i = 1, 2, \ldots\};\ \tilde{x}_i$'s are i.i.d.

$S = \{\tilde{S}_i,\ i = 0, 1, 2, \ldots\}; \tilde{S}_i$'s are general distributed

$N = \{\tilde{n}(t),\ t \geq 0\}; \longrightarrow$ called renewal process

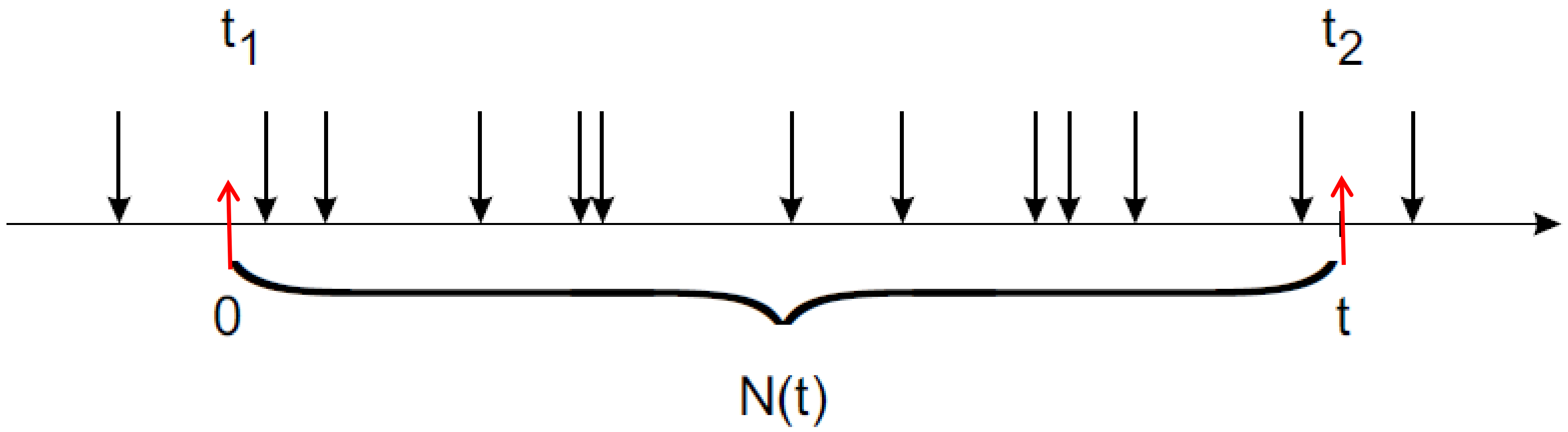
Poisson Process: $X = \{\tilde{x}_i,\ i = 1, 2, \ldots\};\ \tilde{x}_i$'s are iid exponential distributed

$S = \{\tilde{S}_i,\ i = 0, 1, 2, \ldots\}; \tilde{S}_i$'s are Erlang distributed

$N = \{\tilde{n}(t),\ t \geq 0\}; \longrightarrow$ called Poisson process

Poisson process

- Poisson process is one of the most important models used in queueing theory.
  - Often the arrival process of customers can be described by a Poisson process.
  - In teletraffic theory the "customers" may be calls or packets.
  - Poisson process is a viable model when the calls or packets originate from a large population of independent users.
- In the following it is instructive to think that the Poisson process we consider represents discrete arrivals (of e.g. calls or packets).



Poisson Arrival Model

- A Poisson process is a sequence of events "randomly spaced in time"
- For example, customers arriving at a bank and Geiger counter clicks are similar to packets arriving at a buffer
- The rate λ of a Poisson process is the average number of events per unit time (over a long time)

Poisson Process

- Mathematically the process is described by the so called *counter process* N_t or *N(t)*.
- The counter tells the number of arrivals that have occurred in the interval (0, t) or, more generally, in the interval (t1, t2).

$$\begin{cases} N(t) = \text{number of arrivals in the interval } (0,t) & \text{(the stochastic process we consider)} \\ N(t_1,t_2) = \text{number of arrival in the interval } (t_1,t_2) & \text{(the increment process } N(t_2)-N(t_1)) \end{cases}$$

- A Poisson process can be characterized in different ways:
  - Process of independent increments
  - Pure birth process
    - The arrival intensity (mean arrival rate; probability of arrival per time unit)
  - The “most random” process with a given intensity λ

Properties of a Poisson Process

- Properties of a Poisson process
  - For a time interval [0, t] , the probability of *n* arrivals in *t* units of time is

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

  - For two disjoint (non overlapping) intervals (t1, t2) and (t3, t4), (i.e. , t1 < t2 < t3 < t4), the number of arrivals in (t1, t2) is *independent* of arrivals in (t3, t4)

Counting Processes

- A stochastic process $N = \{\tilde{n}(t), t \geq 0\}$ is said to be a *counting process* if $\tilde{n}(t)$ represents the total number of "events" that have occurred up to time t.
- From the definition, we see that for a counting process $\tilde{n}(t)$ must satisfy:

1. $\tilde{n}(t) \geq 0$.
2. $\tilde{n}(t)$ is integer valued.
3. If $s < t$, then $\tilde{n}(s) \leq \tilde{n}(t)$.
4. For $s < t$, $\tilde{n}(t) - \tilde{n}(s)$ equals the number of events that have occurred in the interval $(s, t]$.

Definition 1: Poisson Processes

- The counting process $N = \{\tilde{n}(t)\,,\ t \geq 0\}$ is a *Poisson process* with rate λ ($\lambda > 0$), if:

1. $\tilde{n}(0) = 0$

是指任兩段不重疊的區間內的事件發生次數互不相干

2. Independent increments | relaxed ⇒ Modulated Poisson Process |

$$P[\tilde{n}(t) - \tilde{n}(s) = k_1 | \tilde{n}(r) = k_2,\ r \leq s < t] = P[\tilde{n}(t) - \tilde{n}(s) = k_1]$$

3. Stationary increments | relaxed ⇒ Non-homogeneous Poisson Process |

$$P[\tilde{n}(t+s) - \tilde{n}(t) = k] = P[\tilde{n}(l+s) - \tilde{n}(l) = k]$$

是指某個區間內事件發生次數的機率分配只跟那段區間的長度有關。

4. Single arrival | relaxed ⇒ Compound Poisson Process |

$$P[\tilde{n}(h) = 1] = \lambda h + o(h)$$
$$P[\tilde{n}(h) \geq 2] = o(h)$$

在極短或很小的區域，發生超過一次事件的情況 微乎其微，亦即將時間或區域細分至極小單位， 則事件不是只出現一次，就是不出現。

Definition 2: Poisson Processes

- The counting process $N = \{\tilde{n}(t),\ t \geq 0\}$ is a *Poisson process* with rate λ ($\lambda > 0$), if:

1. $\tilde{n}(0) = 0$
2. Independent increments
3. The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s,\ t \geq 0$

$$P[\tilde{n}(t+s) - \tilde{n}(s) = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, \ldots$$

Theorem: Definitions 1 and 2 are equivalent.

- **Proof.** We show that Definition 1 implies Definition 2. To start, fix $u \geq 0$ and let

$$g(t) = E[e^{-u\tilde{n}(t)}]$$

We derive a differential equation for $g(t)$ as follows:

$$\begin{aligned} g(t+h) &= E[e^{-u\tilde{n}(t+h)}] \\ &= E\left\{e^{-u\tilde{n}(t)}e^{-u[\tilde{n}(t+h)-\tilde{n}(t)]}\right\} \\ &= E\left[e^{-u\tilde{n}(t)}\right]E\left\{e^{-u[\tilde{n}(t+h)-\tilde{n}(t)]}\right\} \quad \text{by independent increments} \\ &= g(t)E\left[e^{-u\tilde{n}(h)}\right] \quad \text{by stationary increments} \qquad (1) \end{aligned}$$

Theorem: Definitions 1 and 2 are equivalent.

Conditioning on whether $\tilde{n}(t) = 0$ or $\tilde{n}(t) = 1$ or $\tilde{n}(t) \geq 2$ yields

$$\begin{aligned} E\left[e^{-u\tilde{n}(h)}\right] &= \boxed{1 - \lambda h + o(h)} + \boxed{e^{-u}(\lambda h + o(h))} + \boxed{o(h)} \\ &= 1 - \lambda h + e^{-u}\lambda h + o(h) \qquad (2) \end{aligned}$$

From (1) and (2), we obtain that

$$g(t+h) = g(t)(1 - \lambda h + e^{-u}\lambda h) + o(h)$$

implying that

differential (微分) ←

$$\frac{g(t+h) - g(t)}{h} = g(t)\lambda(e^{-u} - 1) + \frac{o(h)}{h}$$

Theorem: Definitions 1 and 2 are equivalent.

Letting $h \longrightarrow 0$ gives

$$g'(t) = g(t)\lambda(e^{-u} - 1)$$

or, equivalently,

$$\frac{g'(t)}{g(t)} = \lambda(e^{-u} - 1)$$

Integrating, and using $g(0) = 1$, shows that

$$\log(g(t)) = \lambda t(e^{-u} - 1)$$

or

$$g(t) = e^{\lambda t(e^{-u}-1)} \quad \longrightarrow \quad \text{the Laplace transform of a Poisson r. v.}$$

Since $g(t)$ is also the Laplace transform of $\tilde{n}(t)$, $\tilde{n}(t)$ is a Poisson r. v.

Interarrival Times of Poisson Process

- Interarrival times of a Poisson process
  - We pick an arbitrary starting point t_0 in time. Let T_1 be the time until the next arrival. We have

    $$P(T_1 > t_0) = P_0(t) = e^{-\lambda t}$$

  - Thus the cumulative distribution function of T_1 is given by

    $$FT_1(t) = P(T_1 \leq t) = 1 - e^{-\lambda t}$$

  - The pdf of T_1 is given by $f(x) = \frac{dF_X(x)}{dx}$

    $$fT_1(t) = \lambda e^{-\lambda t}$$

  - Therefore, T_1 has an exponential distribution with mean rate λ

$$\int e^x dx = e^x + C$$

The Inter-Arrival Time Distribution

- **Theorem.** Poisson Processes have exponential inter-arrival time distribution, i.e., $\{\widetilde{x_n}, n = 1, 2, \ldots\}$ are i.i.d and exponentially distributed with parameter λ (i.e., mean inter-arrival time = $1/\lambda$).

Proof.

$$\tilde{x}_1 \quad : \quad P(\tilde{x}_1 > t) = P(\tilde{n}(t) = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

$$\therefore \tilde{x}_1 \sim e(t; \lambda)$$

$$\tilde{x}_2 \quad : \quad P(\tilde{x}_2 > t | \tilde{x}_1 = s)$$

$$= P\{0 \text{ arrivals in } (s, s+t] | \tilde{x}_1 = s\}$$

$$= P\{0 \text{ arrivals in } (s, s+t]\} (\text{by independent increment})$$

$$= P\{0 \text{ arrivals in } (0, t]\} (\text{by stationary increment})$$

$$= e^{-\lambda t} \quad \therefore \tilde{x}_2 \text{ is independent of } \tilde{x}_1 \text{ and } \tilde{x}_2 \sim exp(t; \lambda).$$

$\Rightarrow$ The procedure repeats for the rest of $\tilde{x}_i$'s.

The Arrival Time Distribution of the *n*th Event

- **Theorem.** The arrival time of the n_{th} event, $\widetilde{S_n}$(also called the waiting time until the *n*th event), is *Erlang* distributed with parameter (*n, λ*).

Proof. Method 1 :

$$\because \quad P[\tilde{S}_n \le t] = P[\tilde{n}(t) \ge n] = \sum_{k=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

$$\therefore \quad f_{\tilde{S}_n}(t) = \boxed{\frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}} \text{ (exercise)}$$

→ Erlang distribution

The Erlang distribution with shape parameter k=1 simplifies to the exponential distribution.

Method 2 :

$$\begin{aligned} f_{\tilde{S}_n}(t)dt &= dF_{\tilde{S}_n}(t) = P[t < \tilde{S}_n < t + dt] \\ &= P\{n-1 \text{ arrivals in } (0,t] \text{ and } 1 \text{ arrival in } (t, t+dt)\} + o(dt) \\ &= P[\tilde{n}(t) = n-1 \text{ and } 1 \text{ arrival in } (t, t+dt)] + o(dt) \\ &= P[\tilde{n}(t) = n-1]P[1 \text{ arrival in } (t, t+dt)] + o(dt)(why?) \end{aligned}$$

independent increments

The Arrival Time Distribution of the *n*th Event

$$
\begin{aligned}
&= \quad \frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}\lambda dt + o(dt) \\
&\therefore \quad \lim_{dt\to 0} \frac{f_{\tilde{S}_n}(t)dt}{dt} = f_{\tilde{S}_n}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}
\end{aligned}
$$

Conditional Distribution of the Arrival Times

- **Theorem.** Given that $\tilde{n}(t)$ = *n*, the *n* arrival times $\widetilde{S_1}, \widetilde{S_2}, \ldots, \widetilde{S_n}$ have the same distribution as the order statistics corresponding to *n* i.i.d. uniformly distributed random variables from (0, *t*).

Order Statistics. Let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n$ be n i.i.d. continuous random variables having common pdf f. Define $\tilde{x}_{(k)}$ as the k_{th} smallest value among all $\tilde{x}_i$'s, i.e., $\tilde{x}_{(1)} \leq \tilde{x}_{(2)} \leq \tilde{x}_{(3)} \leq \ldots \leq \tilde{x}_{(n)}$, then $\tilde{x}_{(1)}, \ldots, \tilde{x}_{(n)}$ are known as the "order statistics" corresponding to random variables $\tilde{x}_1, \ldots, \tilde{x}_n$. We have that the joint pdf of $\tilde{x}_{(1)}, \tilde{x}_{(2)}, \ldots, \tilde{x}_{(n)}$ is

$$f_{\tilde{x}_{(1)}, \tilde{x}_{(2)}, \ldots, \tilde{x}_{(n)}}(x_1, x_2, \ldots, x_n) = n! f(x_1) f(x_2) \ldots f(x_n),$$

where $x_1 < x_2 < \ldots < x_n$ (check the textbook [Ross]).

Conditional Distribution of the Arrival Times

Proof. Let $0 < t_1 < t_2 < \ldots < t_{n+1} = t$ and let h_i be small enough so that $t_i + h_i < t_{i+1}\ ,i = 1, \ldots, n$.

$$
\begin{aligned}
\because \quad & P[t_i < \tilde{S}_i < t_i + h_i,\ i = 1, \ldots, n | \tilde{n}(t) = n] \\
= \quad & \frac{P\left(\begin{array}{c}\text{exactly one arrival in each } [t_i, t_i + h_i] \\ i = 1, 2, \ldots, n,\ \text{ and no arrival elsewhere in } [0, t]\end{array}\right)}{P[\tilde{n}(t) = n]} \\
= \quad & \frac{(e^{-\lambda h_1}\lambda h_1)(e^{-\lambda h_2}\lambda h_2)\ldots(e^{-\lambda h_n}\lambda h_n)(e^{-\lambda(t-h_1-h_2\ldots-h_n)})}{e^{-\lambda t}(\lambda t)^n/n!} \\
= \quad & \frac{n!(h_1 h_2 h_3 \ldots h_n)}{t^n} \\
\therefore \quad & \frac{P[t_i < \tilde{S}_i < t_i + h_i,\ i = 1, \ldots, n | \tilde{n}(t) = n]}{h_1 h_2 \ldots h_n} = \frac{n!}{t^n}
\end{aligned}
$$

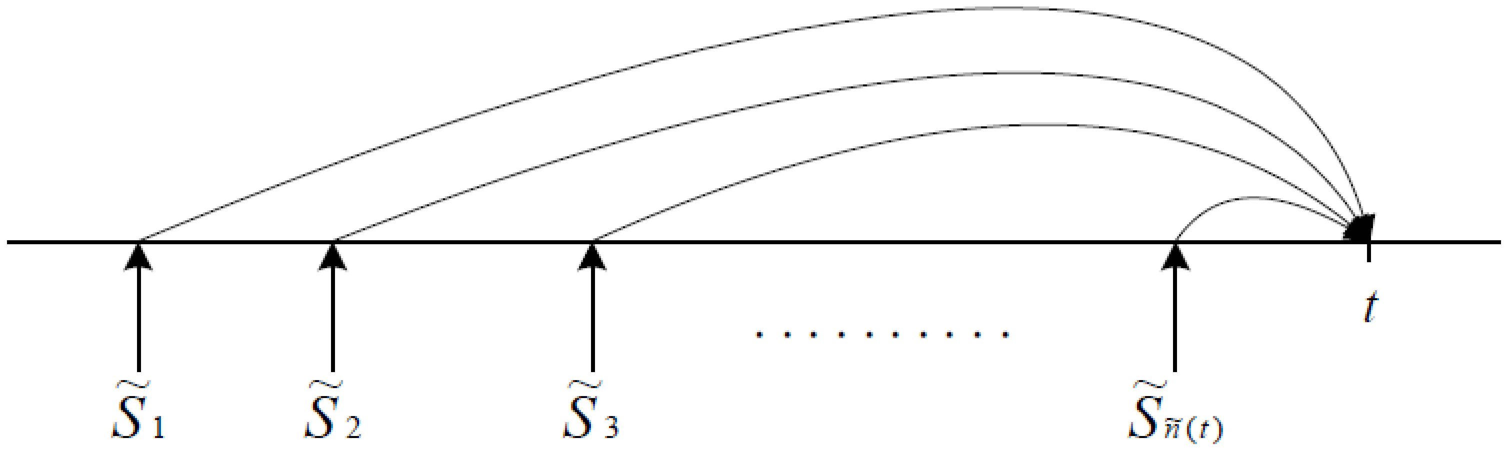
Conditional Distribution of the Arrival Times

Taking $\lim_{h_i \to 0, i=1,\ldots,n} (\qquad)$, then

$$f_{\tilde{S}_1,\tilde{S}_2,\ldots,\tilde{S}_n|\tilde{n}(t)}(t_1,t_2,\ldots,t_n|n) = \frac{n!}{t^n},\ 0 < t_1 < t_2 < \ldots < t_n.$$

Conditional Distribution of the Arrival Times

Example (see Ref [Ross], Ex. 2.3(A) p.68). Suppose that travellers arrive at a train depot in accordance with a Poisson process with rate λ. If the train departs at time t, what is the expected sum of the waiting times of travellers arriving in $(0,t)$? That is, $E[\sum_{i=1}^{\tilde{n}(t)}(t-\tilde{S}_i)] = ?$



Conditional Distribution of the Arrival Times

Answer. Conditioning on $\tilde{n}(t) = n$ yields

$$E[\sum_{i=1}^{\tilde{n}(t)}(t - \tilde{S}_i)|\tilde{n}(t) = n] = nt - E[\sum_{i=1}^{n} \tilde{S}_i]$$

$$= nt - E[\sum_{i=1}^{n} \tilde{u}_{(i)}] \qquad \text{(by the theorem)}$$

$$= nt - E[\sum_{i=1}^{n} \tilde{u}_i] \qquad (\because \sum_{i=1}^{n} \tilde{u}_{(i)} = \sum_{i=1}^{n} \tilde{u}_i)$$

$$= nt - \frac{t}{2} \cdot n = \frac{nt}{2} \qquad (\because E[\tilde{u}_i] = \frac{t}{2})$$

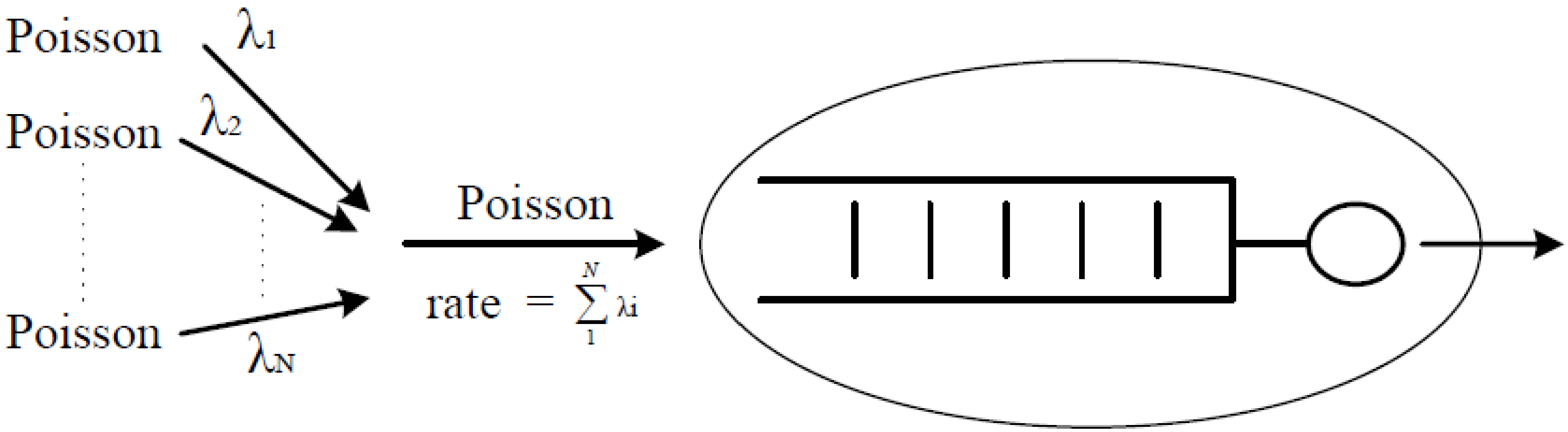
To find $E[\sum_{i=1}^{\tilde{n}(t)}(t - \tilde{S}_i)]$, we should take another expectation

$$\therefore \quad E[\sum_{i=1}^{\tilde{n}(t)}(t - \tilde{S}_i)] = \frac{t}{2} \cdot \underbrace{E[\tilde{n}(t)]}_{=\lambda t} = \frac{\lambda t^2}{2}$$

Superposition of Independent Poisson Processes

- **Theorem.** Superposition of independent Poisson Processes

$(\lambda_i, i = 1, \ldots, N)$, is also a Poisson process with rate $\sum_{1}^{N} \lambda_i$.



Decomposition of a Poisson Process

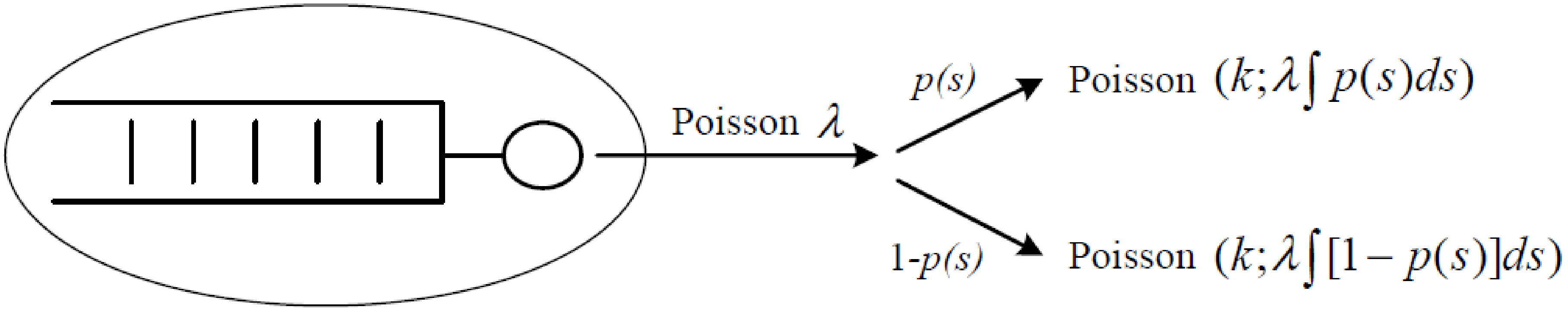
Theorem.

- Given a Poisson process $N = \{\tilde{n}(t), t \geq 0\}$;
- If $\tilde{n}_i(t)$ represents the number of type-i events that occur by time $t, i = 1, 2$;
- Arrival occurring at time s is a type-1 arrival with probability $p(s)$, and type-2 arrival with probability $1 - p(s)$

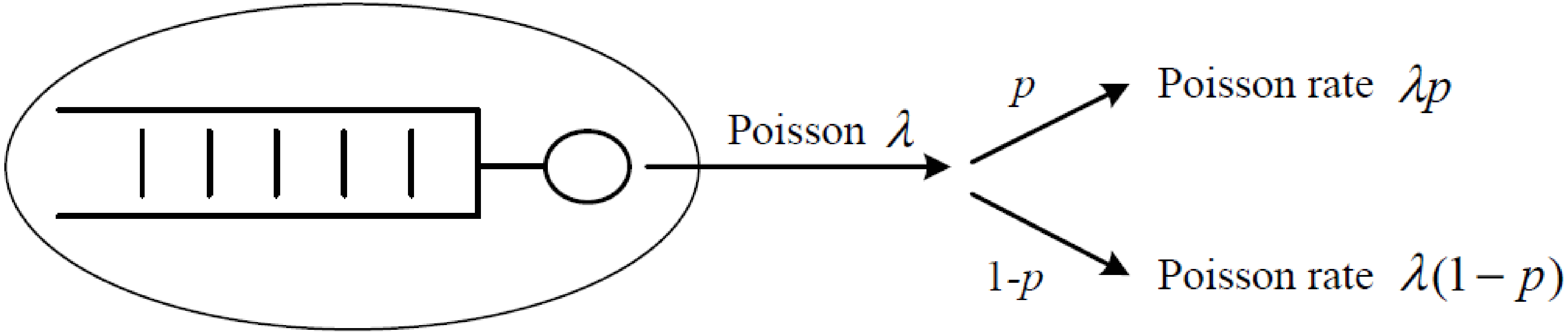
$\Downarrow$then

- $\tilde{n}_1, \tilde{n}_2$ are independent,
- $\tilde{n}_1(t) \sim P(k; \lambda t p)$, and
- $\tilde{n}_2(t) \sim P(k; \lambda t(1-p))$, where $p = \frac{1}{t}\int_0^t p(s)ds$

Decomposition of a Poisson Process



special case: If $p(s) = p$ is constant, then



Decomposition of a Poisson Process

Proof. It is to prove that, for fixed time t,

$$
\begin{aligned}
P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m] &= P[\tilde{n}_1(t) = n] P[\tilde{n}_2(t) = m] \\
&= \frac{e^{-\lambda p t}(\lambda p t)^n}{n!} \cdot \frac{e^{-\lambda(1-p)t}[\lambda(1-p)t]^m}{m!}
\end{aligned}
$$

. . . .

$$
\begin{aligned}
& P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m] \\
= \; & \sum_{k=0}^{\infty} P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m | \tilde{n}_1(t) + \tilde{n}_2(t) = k] \cdot P[\tilde{n}_1(t) + \tilde{n}_2(t) = k] \\
= \; & P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m | \tilde{n}_1(t) + \tilde{n}_2(t) = n + m] \cdot P[\tilde{n}_1(t) + \tilde{n}_2(t) = n + m]
\end{aligned}
$$

Decomposition of a Poisson Process

- From the "condition distribution of the arrival times", any event occurs at some time that is uniformly distributed, and is independent of other events.
- Consider that only one arrival occurs in the interval [0, *t*]:

$$
\begin{aligned}
& P[\text{type - 1 arrival}|\tilde{n}(t)=1] \\
= \; & \int_0^t P[\text{type - 1 arrival}|\text{arrival time } \tilde{S}_1 = s, \tilde{n}(t)=1] \\
& \qquad\qquad\qquad\qquad \times f_{\tilde{S}_1|\tilde{n}(t)}(s|\tilde{n}(t)=1)ds \\
= \; & \int_0^t P(s)\cdot\frac{1}{t}ds = \frac{1}{t}\int_0^t P(s)ds = p
\end{aligned}
$$

Decomposition of a Poisson Process

$$\therefore \quad P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m]$$

$$= \quad P[\tilde{n}_1(t) = n, \tilde{n}_2(t) = m | \tilde{n}_1(t) + \tilde{n}_2(t) = n + m] \cdot P[\tilde{n}_1(t) + \tilde{n}_2(t) = n + m]$$

$$= \quad \binom{n+m}{n} p^n (1-p)^m \cdot \frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!}$$

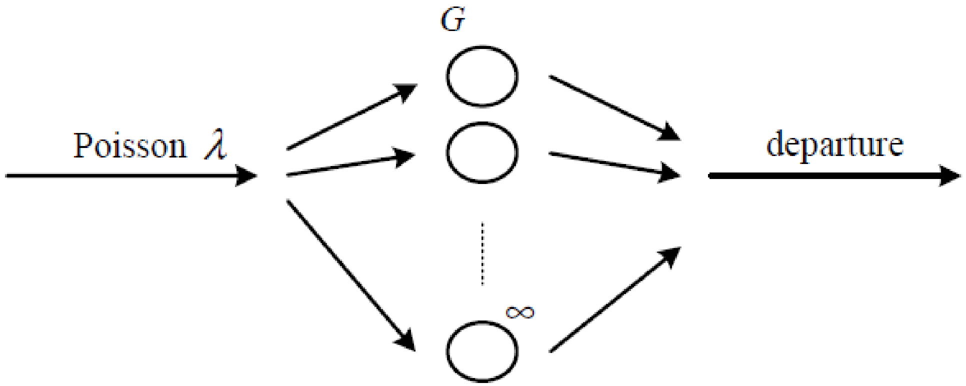
Binomial Distribution

$$= \quad \frac{(n+m)!}{n!m!} p^n (1-p)^m \cdot \frac{e^{-\lambda t}(\lambda t)^{n+m}}{(n+m)!}$$

$$= \quad \frac{e^{-\lambda p t}(\lambda p t)^n}{n!} \cdot \frac{e^{-\lambda(1-p)t}[\lambda(1-p)t]^m}{m!}$$

Decomposition of a Poisson Process

- **Example** (An Infinite Server Queue, textbook [Ross]).



- $G_{\tilde{s}}(t) = P(\tilde{S} \leq t)$, where $\tilde{S}$ = service time
- $G_{\tilde{s}}(t)$ is independent of each other and of the arrival process
- $\tilde{n}_1(t)$: the number of customers which have left before t;
- $\tilde{n}_2(t)$: the number of customers which are still in the system at time t;

$\Rightarrow \tilde{n}_1(t) \sim ?$ and $\tilde{n}_2(t) \sim ?$

Decomposition of a Poisson Process

- **Answer.**
- $\widetilde{n_1}(t)$: the number of type-1 customers
- $\widetilde{n_2}(t)$: the number of type-2 customers

$$
\begin{aligned}
\text{type-1:} \quad P(s) &= P(\text{finish before } t) \\
&= P(\tilde{S} \leq t - s) = G_{\tilde{s}}(t - s) \\
\text{type-2:} \quad 1 - P(s) &= \bar{G}_{\tilde{s}}(t - s)
\end{aligned}
$$

$$
\begin{aligned}
\therefore \qquad \tilde{n}_1(t) &\sim P\left(k; \lambda t \cdot \frac{1}{t} \int_0^t G_{\tilde{s}}(t - s) ds\right) \\
\tilde{n}_2(t) &\sim P\left(k; \lambda t \cdot \frac{1}{t} \int_0^t \bar{G}_{\tilde{s}}(t - s) ds\right)
\end{aligned}
$$

Decomposition of a Poisson Process

$$\therefore \quad E[\tilde{n}_1(t)] = \lambda t \cdot \frac{1}{t}\int_0^t G(t-s)ds$$

$$= \lambda \int_t^0 G(y)(-dy) \qquad t - s = y$$

$$s = t - y$$

$$= \lambda \int_0^t G(y)dy$$

As $t \to \infty$, we have

$$\lim_{t\to\infty} E[\tilde{n}_2(t)] = \lambda \int_0^t \bar{G}(y)dy = \lambda E[\tilde{S}] \quad \text{(Little's formula)}$$