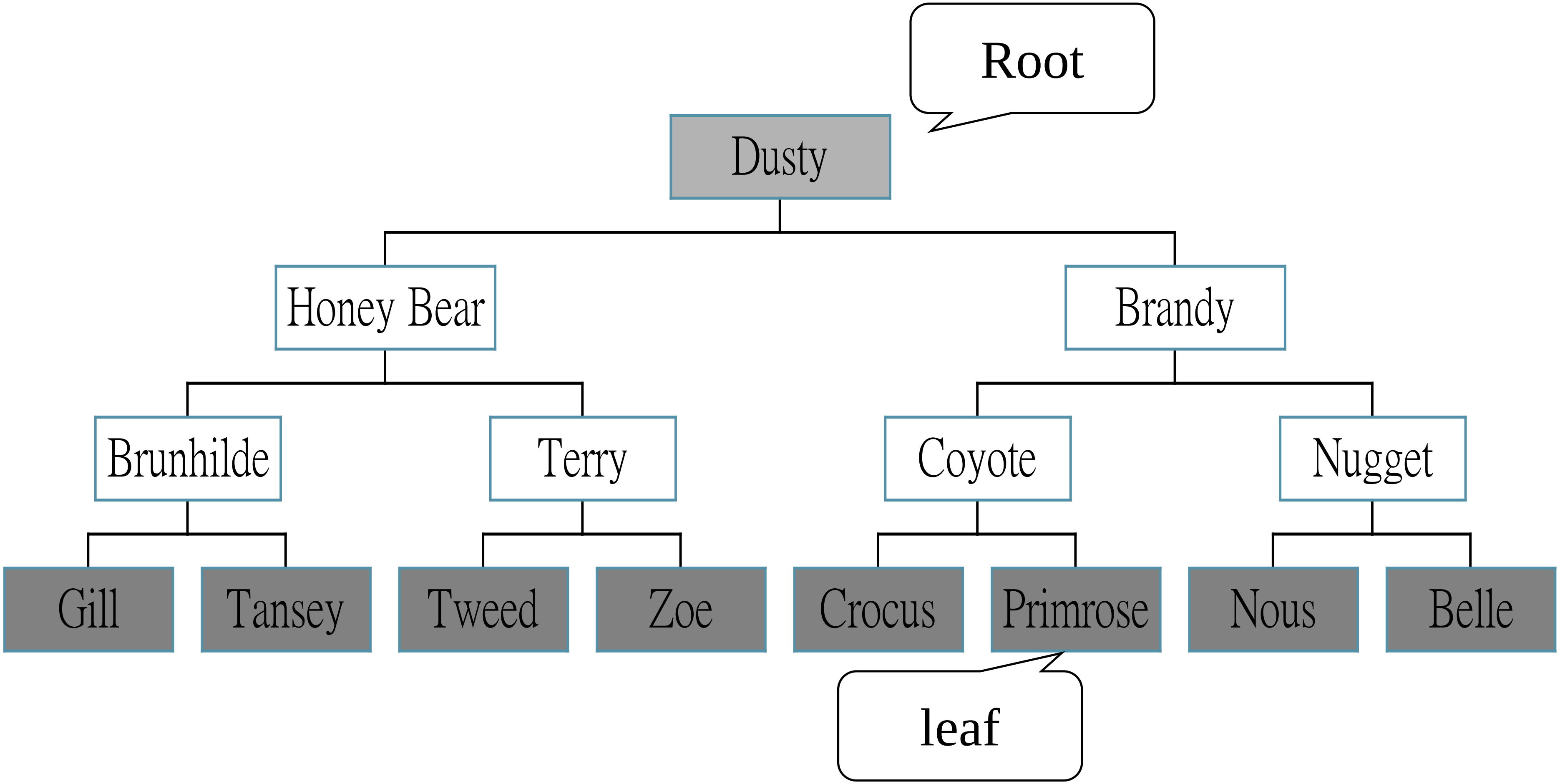
CHAPTER 5

Trees

All the programs in this file are selected from

Ellis Horowitz, Sartaj Sahni, and Susan Anderson-Freed
"Fundamentals of Data Structures in C",

Trees

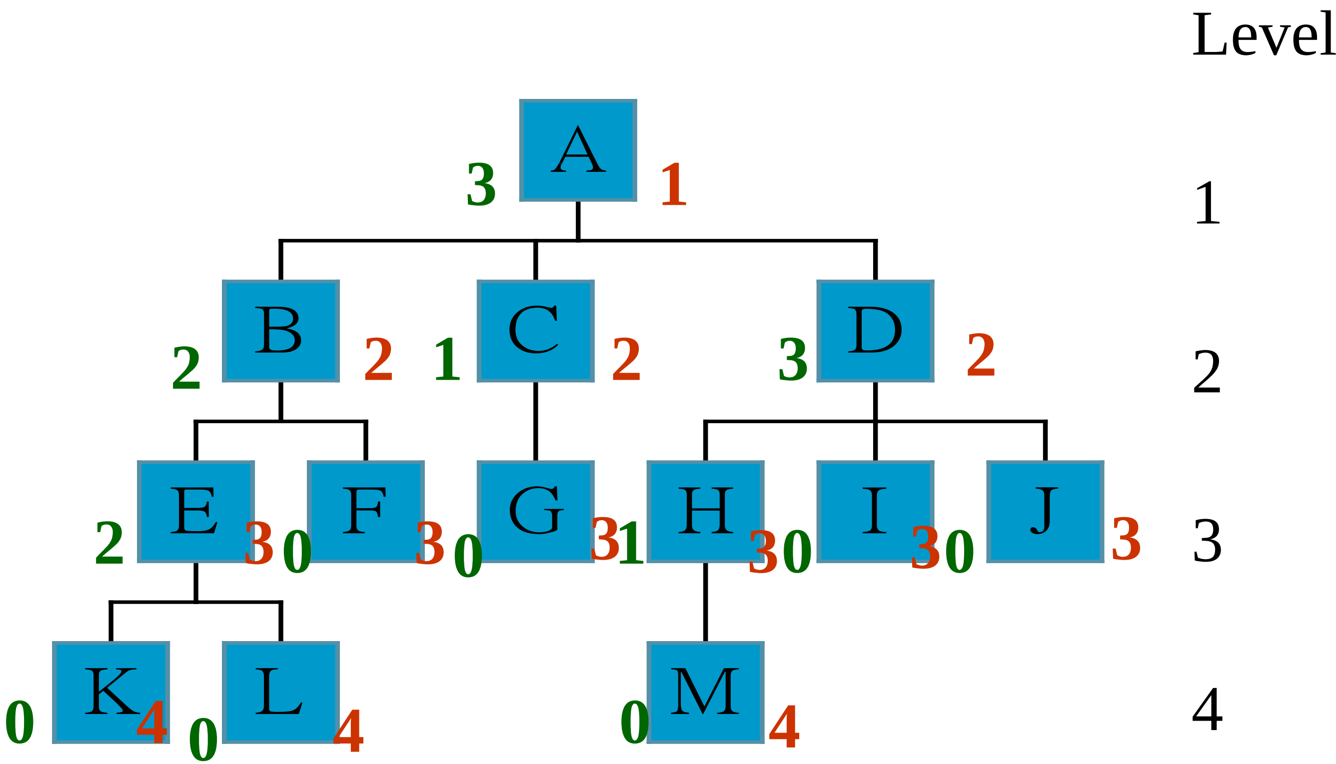

Root
Dusty
Honey Bear
Brandy
Brunhilde
Terry
Coyote
Nugget
Gill
Tansey
Tweed
Zoe
Crocus
Primrose
Nous
Belle
leaf

Definition of Tree

- A tree is a finite set of one or more nodes such that:
  - There is a specially designated node called the **root**.
  - The remaining nodes are partitioned into n>=0 disjoint sets T1, ..., Tn, where each of these sets is a tree.
  - We call T1, ..., Tn the subtrees of the root.

Level and Depth

1. node (13)
2. leaf (terminal)
3. nonterminal
4. parent
5. children
6. sibling
7. degree of a tree (3)
8. ancestor
9. level of a node
10. height of a tree (4)



Terminology

- The *degree* of a node is the number of subtrees of the node
  - The degree of A is 3; the degree of C is 1.
- The node with degree 0 is a leaf or terminal node.
- A node that has subtrees is the *parent* of the subtrees.
- These subtrees are the *children* of the node.
- Children of the same parent are *siblings*.
- The *ancestors* of a node are all the nodes along the path from the root to the node.

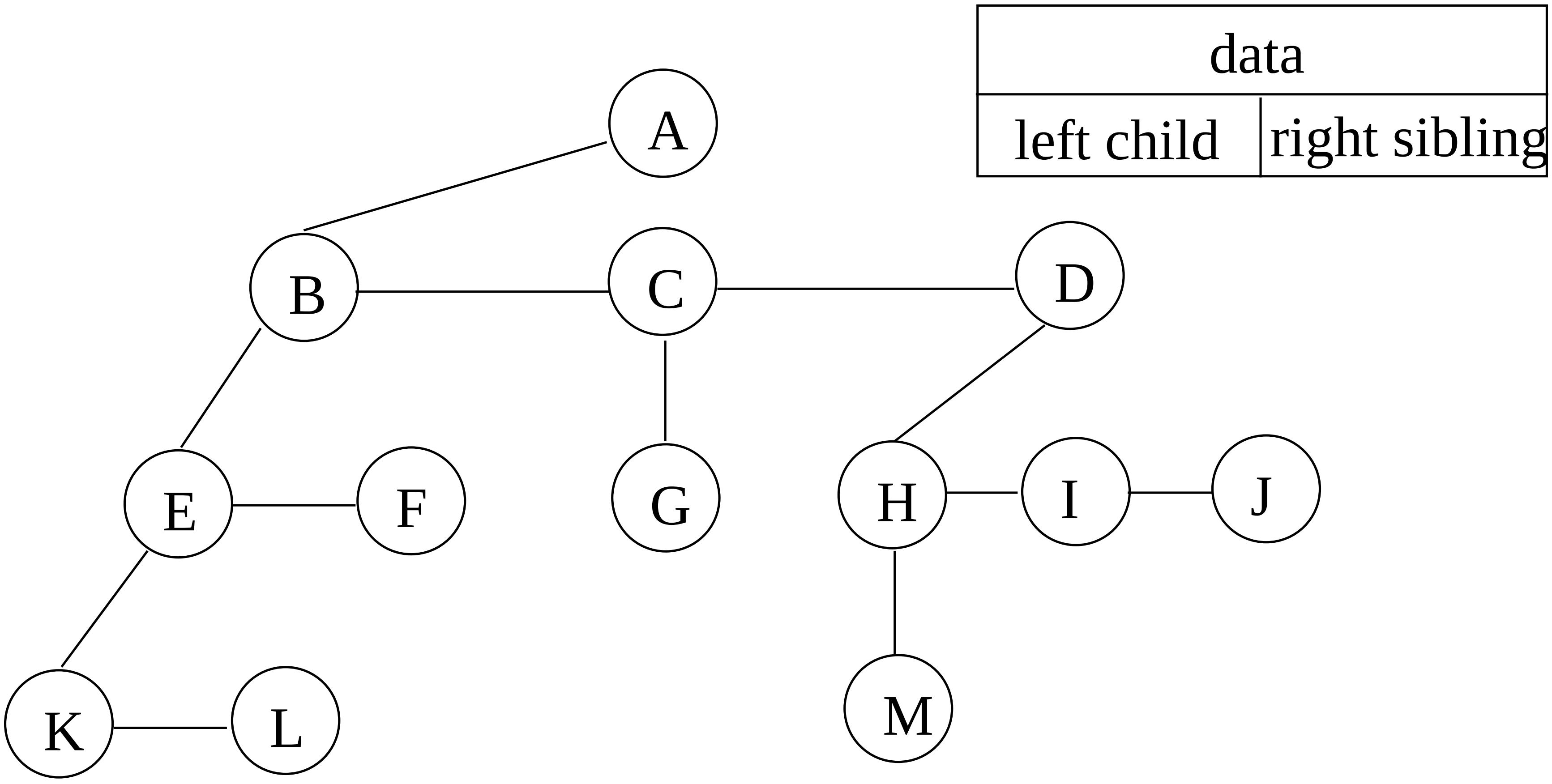
Representation of Trees

- List Representation
  - (A (B (E (K, L), F), C (G), D (H (M), I, J)))
  - The root comes first, followed by a list of sub-trees

data	link 1	link 2	...	link n

How many link fields are needed in such a representation?

Left Child - Right Sibling



Binary Trees

- A binary tree is a finite set of nodes that is either empty or consists of a root and two disjoint binary trees called *the left subtree* and *the right subtree*.
- Any tree can be transformed into binary tree.
  – by left child-right sibling representation
- The left subtree and the right subtree are distinguished.

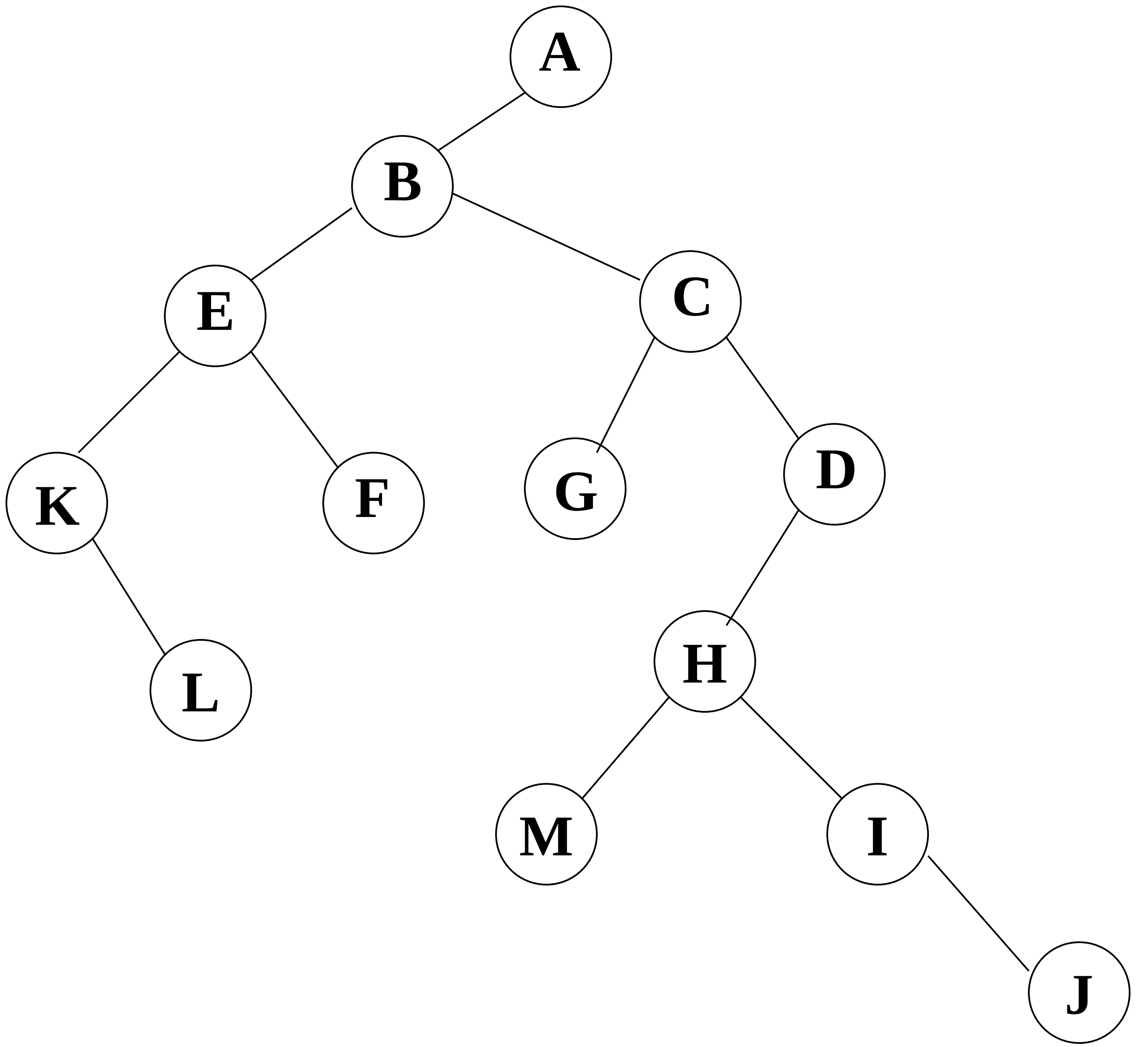


***Figure 5.2** Left child-right child tree representation of a tree

Abstract Data Type Binary_Tree

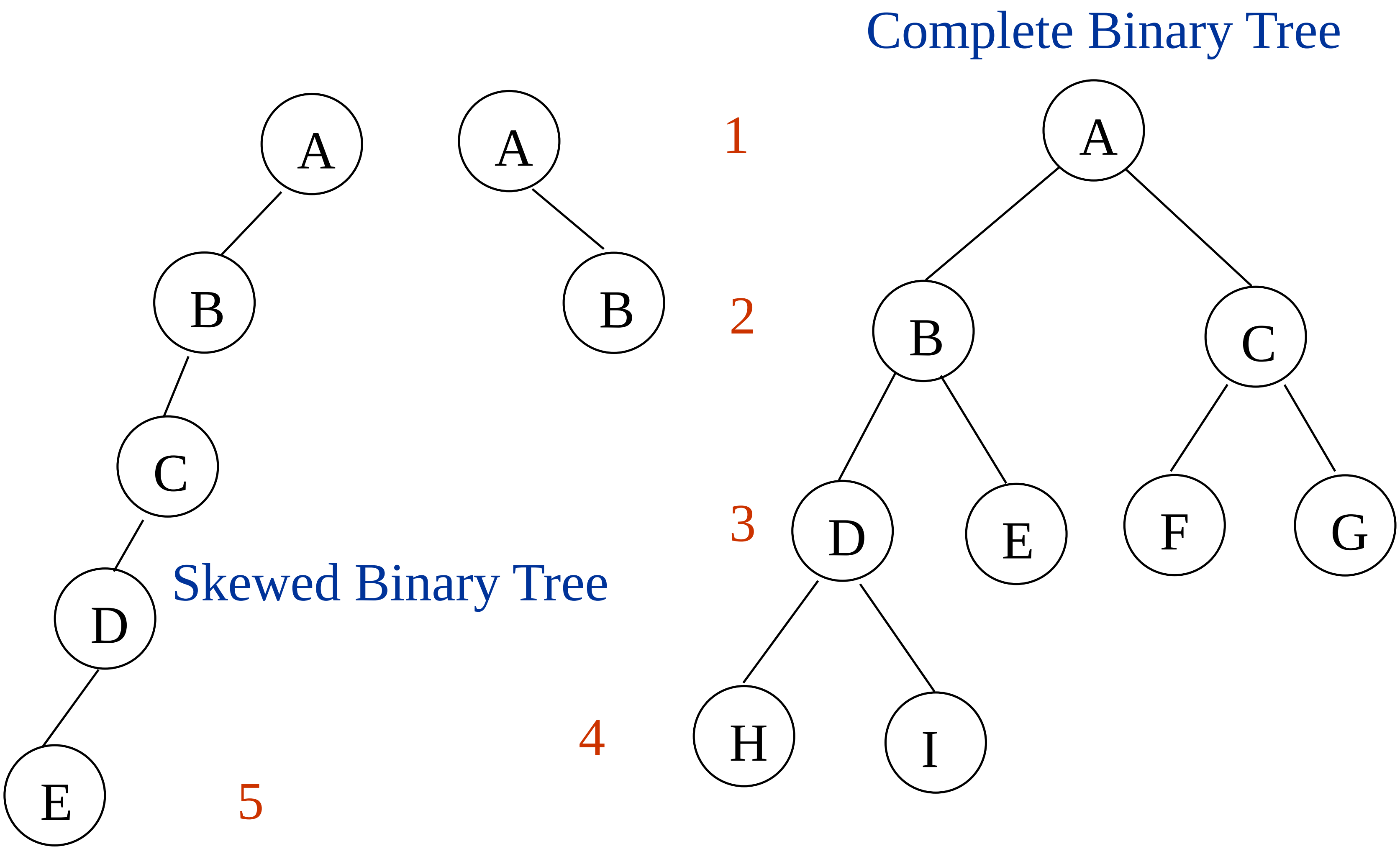
- structure *Binary_Tree* (abbreviated *BinTree*) is
- objects: a finite set of nodes either empty or consisting of a root node, *left Binary_Tree*, and *right Binary_Tree*.
- functions:

  for all *bt*, *bt1*, *bt2* $\in$ *BinTree*, *item* $\in$ *element*
- *Bintree* Create()::= creates an empty binary tree
- *Boolean* IsEmpty(*bt*)::= if (*bt*==empty binary tree) return *TRUE* else return *FALSE*

Abstract Data Type Binary_Tree

- *BinTree* MakeBT(*bt1*, *item*, *bt2*)::= return a binary tree whose left subtree is *bt1*, whose right subtree is *bt2*, and whose root node contains the data *item*
- *Bintree* Lchild(*bt*)::= if (IsEmpty(*bt*)) return error else return the left subtree of *bt*
- *element* Data(*bt*)::= if (IsEmpty(*bt*)) return error else return the data in the root node of *bt*
- *Bintree* Rchild(*bt*)::= if (IsEmpty(*bt*)) return error else return the right subtree of *bt*

Samples of Trees



Maximum Number of Nodes in BT

- The maximum number of nodes on level i of a binary tree is 2^{i-1}, $i>=1$.
- The maximum nubmer of nodes in a binary tree of depth k is 2^k-1, $k>=1$.

Prove by induction.

$$\sum_{i=1}^{k} 2^{i-1} = 2^k - 1$$

pp. 200

Relations between Number of Leaf Nodes and Nodes of Degree 2

- *For any nonempty binary tree, T, if $n0$ is the number of leaf nodes and $n2$ the number of nodes of degree 2, then $n0=n2+1$*

proof:

- Let n and B denote the total number of nodes & branches in T.
- Let $n0$, $n1$, $n2$ represent the nodes with no children, single child, and two children respectively.

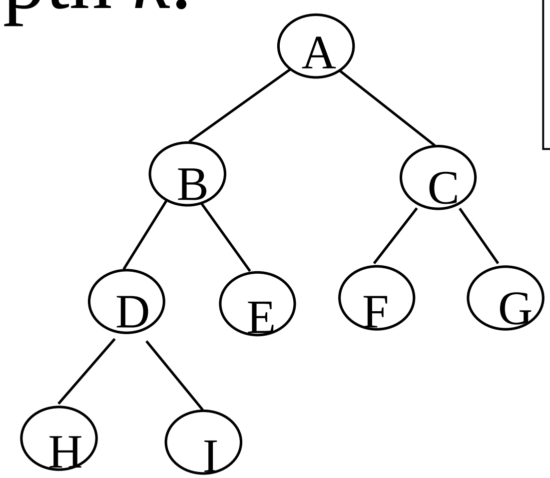
$n0=n2+1$

$n= n0+n1+n2$, $n=B+1$, $n=B+1=n1+2n2+1$,
$n1+2n2+1= n0+n1+n2$ ==> $n0=n2+1$

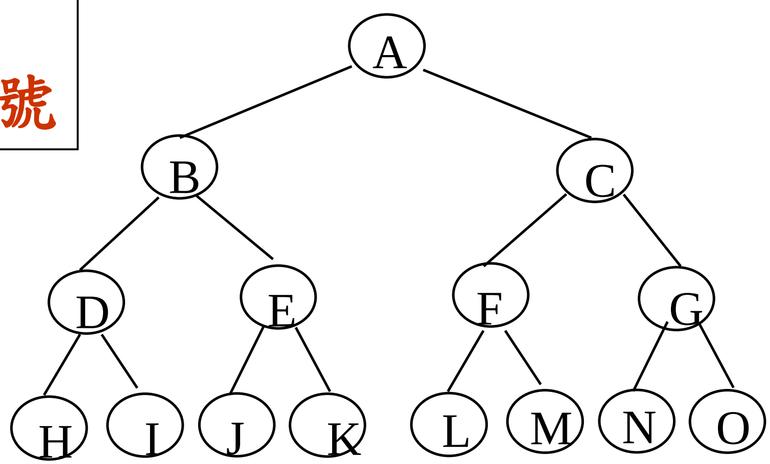
Full BT VS Complete BT

- A full binary tree of depth k is a binary tree of depth k having 2^k-1 nodes, k>=0.
- A binary tree with n nodes and depth k is complete *iff* its nodes correspond to the nodes numbered from 1 to n in the full binary tree of depth k.

由上至下，
由左至右編號



Complete binary tree



Full binary tree of depth 4

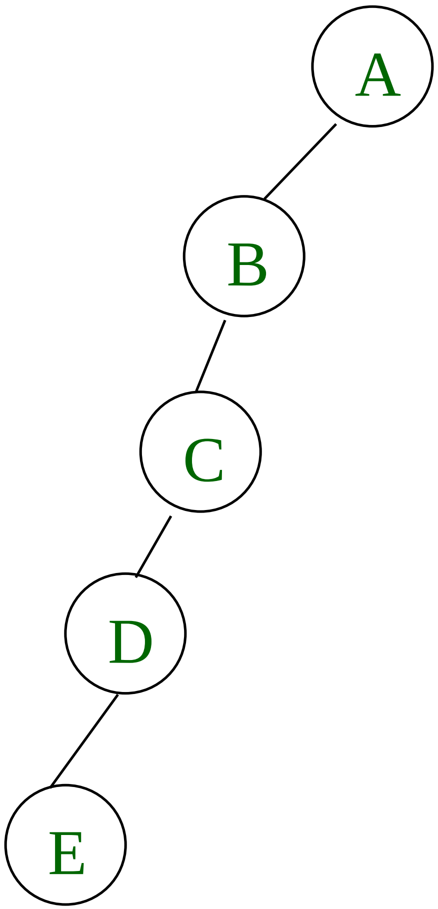
Binary Tree Representations

- If a complete binary tree with n nodes (depth = $\log n + 1$) is represented sequentially, then for any node with index i, $1<=i<=n$, we have:
  - *parent*(i) is at **$i/2$** if $i!=1$. If $i=1$, i is at the root and has no parent.
  - *left_child*(i) ia at **$2i$** if $2i<=n$. If $2i>n$, then i has no left child.
  - *right_child*(i) ia at **$2i+1$** if $2i+1<=n$. If $2i+1>n$, then i has no right child.

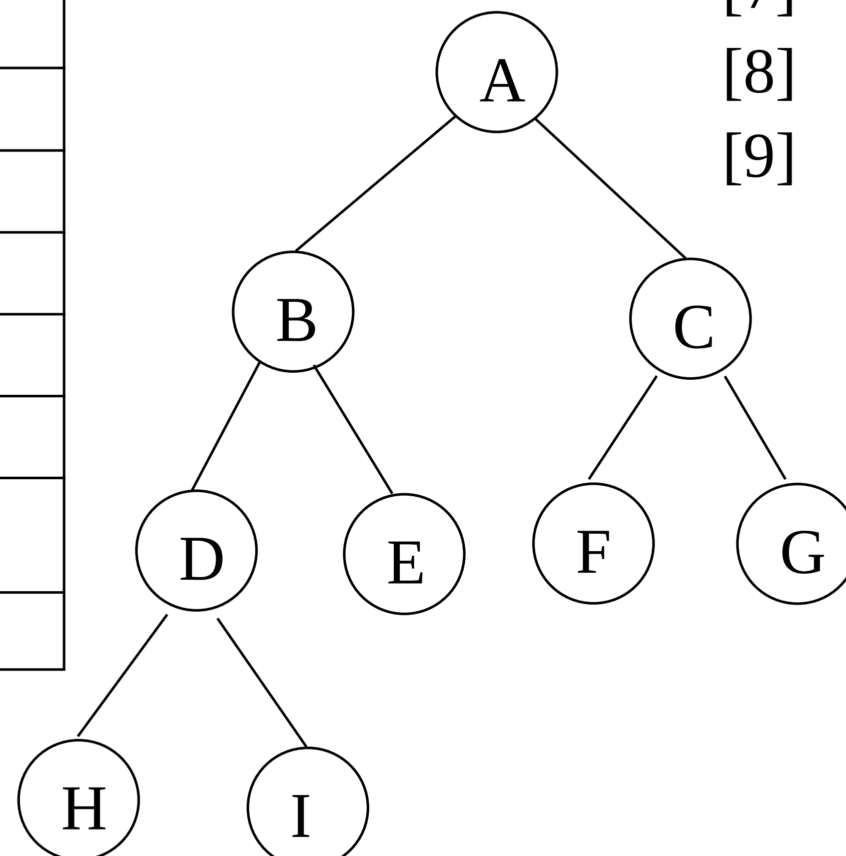
Sequential Representation

(1) waste space

(2) insertion/deletion problem



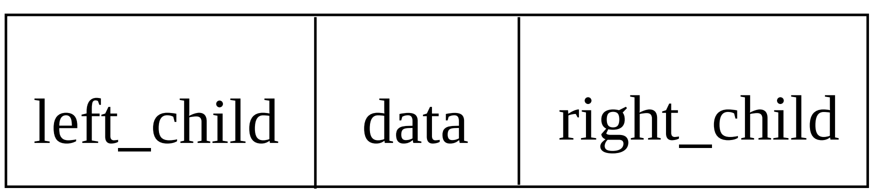
[1]	A
[2]	B
[3]	--
[4]	C
[5]	--
[6]	--
[7]	--
[8]	D
[9]	--
.	.
[16]	E

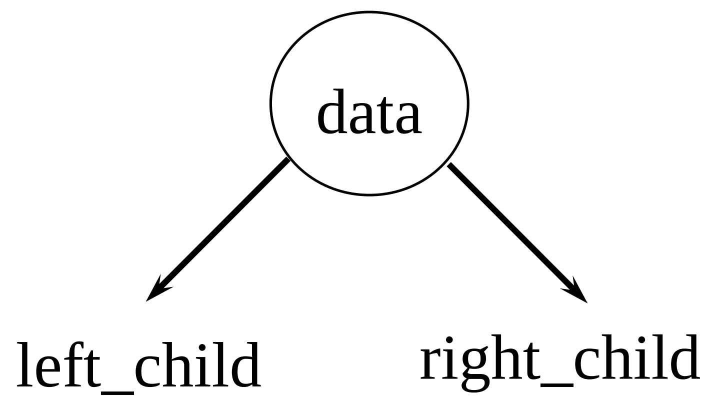


[1]	A
[2]	B
[3]	C
[4]	D
[5]	E
[6]	F
[7]	G
[8]	H
[9]	I

Linked Representation

```
typedef struct node *tree_pointer;
typedef struct node {
 int data;
 tree_pointer left_child, right_child;
};
```

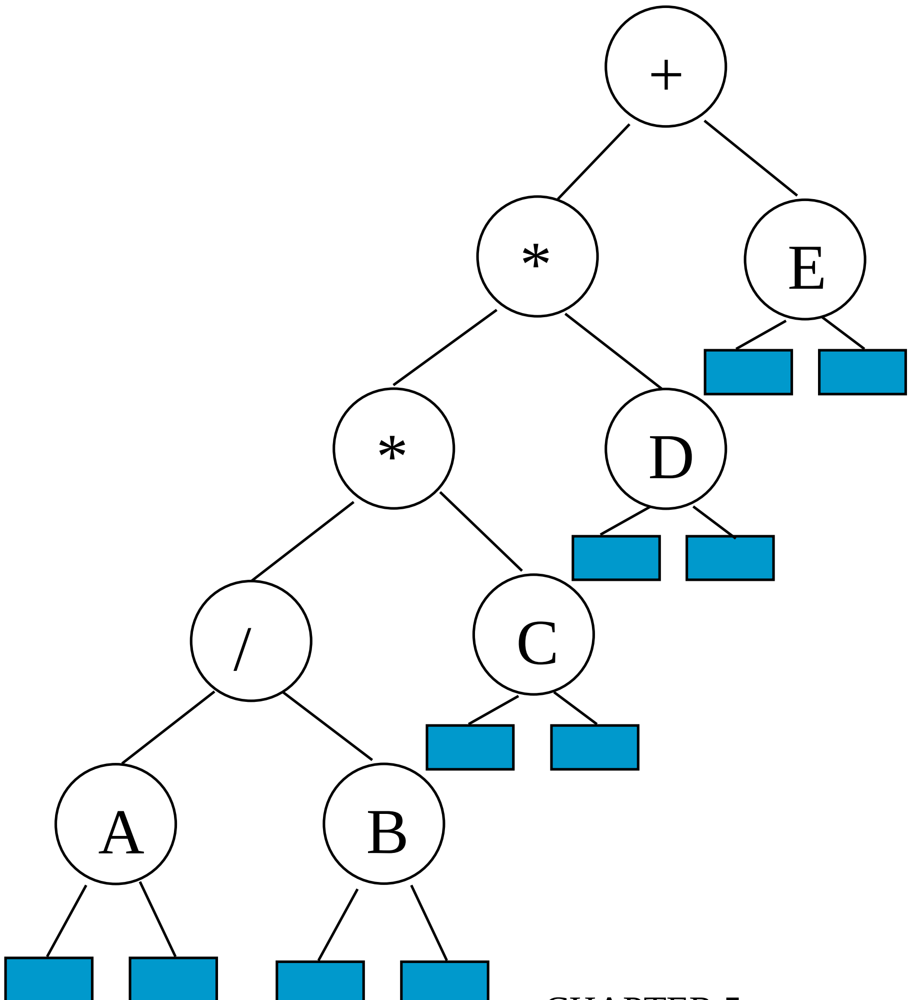




Binary Tree Traversals

- Let L, V, and R stand for moving left, visiting the node, and moving right.
- There are six possible combinations of traversal
  - LVR, LRV, VLR, VRL, RVL, RLV
- Adopt convention that we traverse left before right, only 3 traversals remain
  - LVR, LRV, VLR
  - inorder, postorder, preorder

Arithmetic Expression Using BT



inorder traversal

A / B * C * D + E

infix expression

preorder traversal

+ * * / A B C D E

prefix expression

postorder traversal

A B / C * D * E +

postfix expression

level order traversal

+ * E * D / C A B

Inorder Traversal (recursive version)

```
void inorder(tree_pointer ptr)
/* inorder tree traversal */
{
    if (ptr) {
        inorder(ptr->left_child);
        printf("%d", ptr->data);
        inorder(ptr->right_child);
    }
}
```

A / B * C * D + E

Preorder Traversal (recursive version)

```
void preorder(tree_pointer ptr)
/* preorder tree traversal */
{
    if (ptr) {
        printf("%d", ptr->data);
        preorder(ptr->left_child);
        preorder(ptr->right_child);
    }
}
```

+ * * / A B C D E

Postorder Traversal (recursive version)

```
void postorder(tree_pointer ptr)
/* postorder tree traversal */
{
    if (ptr) {
        postorder(ptr->left_child);
        postorder(ptr->right_child);
        printf("%d", ptr->data);
    }
}
```

A B / C * D * E +

Iterative Inorder Traversal

(using stack)

```
void iterInorder(tree_pointer node)
{
  int top= -1; /* initialize stack */
  tree_pointer stack[MAX_STACK_SIZE];
  for (;;) {
   for (; node; node=node->left_child)
     push(&top, node);/* add to stack */
   node= pop(&top);
                 /* delete from stack */
   if (!node) break; /* empty stack */
   printf("%D", node->data);
   node = node->right_child;
 }
}
```

O(n)

Trace Operations of Inorder Traversal

Call of inorder	Value in root	Action	Call of inorder	Value in root	Action
1	+		11	C	
2	*		12	NULL	
3	*		11	C	printf
4	/		13	NULL	
5	A		2	*	printf
6	NULL		14	D	
5	A	printf	15	NULL	
7	NULL		14	D	printf
4	/	printf	16	NULL	
8	B		1	+	printf
9	NULL		17	E	
8	B	printf	18	NULL	
10	NULL		17	E	printf
3	*	printf	19	NULL	

LVR

Level Order Traversal

(using queue)

```
void levelOrder(tree_pointer ptr)
/* level order tree traversal */
{
  int front = rear = 0;
  tree_pointer queue[MAX_QUEUE_SIZE];
  if (!ptr) return; /* empty queue */
  addq(ptr);
  for (;;) {
    ptr = delete();
```

```
    if (ptr) {
      printf(“%d”, ptr->data);
      if (ptr->left_child)
        addq(ptr->left_child);
      if (ptr->right_child)
        addq(ptr->right_child);
    }
    else break;
  }
}
```

+ * E * D / C A B

Copying Binary Trees

```
tree_pointer copy(tree_pointer original)
{
tree_pointer temp;
if (original) {
  temp=(tree_pointer) malloc(sizeof(node));
  if (IS_FULL(temp)) {
    fprintf(stderr, “the memory is full\n”);
    exit(1);
  }
  temp->left_child=copy(original->left_child);
  temp->right_child=copy(original->right_child);
  temp->data=original->data;
  return temp;
 }
return NULL;
}
```

postorder

Equality of Binary Trees

the same topology and data

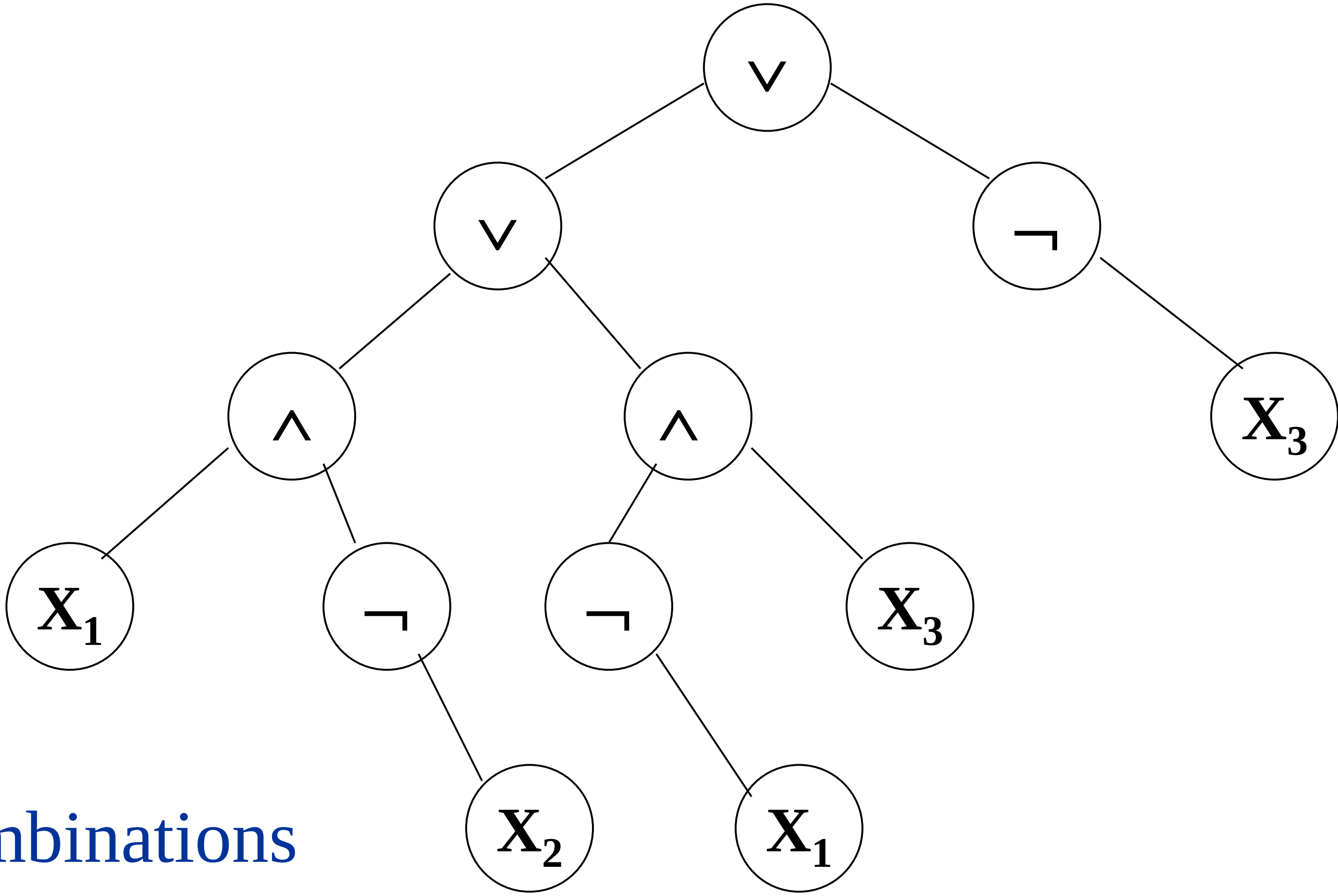
```
int equal(tree_pointer first, tree_pointer second)
{
/* function returns FALSE if the binary trees first
  and second are not equal, otherwise it returns TRUE
  */
  return ((!first && !second) || (first && second &&
       (first->data == second->data) &&
       equal(first->left_child, second->left_child) &&
       equal(first->right_child, second->right_child)))
}
```

Propositional Calculus Expression

- A variable is an expression.
- If x and y are expressions, then $\neg x$, $x \wedge y$, $x \vee y$ are expressions.
- Parentheses can be used to alter the normal order of evaluation ($\neg > \wedge > \vee$).
- Example: $x_1 \vee (x_2 \wedge \neg x_3)$
- satisfiability problem: Is there an assignment to make an expression true?

$$(x_1 \wedge \neg x_2) \vee (\neg x_1 \wedge x_3) \vee \neg x_3$$

(t,t,t)
(t,t,f)
(t,f,t)
(t,f,f)
(f,t,t)
(f,t,f)
(f,f,t)
(f,f,f)



2^n possible combinations for n variables

postorder traversal (postfix evaluation)

LRV

Node Structure

left_child	***data***	***value***	***right_child***

```
typedef emun {not, and, or, true, false } logical;
typedef struct node *tree_pointer;
typedef struct node {
        tree_pointer  left_child;
        logical       data;
        short int     value;
        tree_pointer  right_child;
        } ;
```

First version of satisfiability algorithm

```
for (all 2^n possible combinations) {
    generate the next combination;
    replace the variables by their values;
    evaluate root by traversing it in postorder;
    if (root->value) {
        printf(<combination>);
        return;
    }
}
printf("No satisfiable combination \n");
```

Post-order-eval function

```
void postOrderEval(tree_pointer node)
{
/* modified post order traversal to evaluate a propositional
calculus tree */
    if (node) {
        post_order_eval(node->left_child);
        post_order_eval(node->right_child);
        switch(node->data) {
           case not:  node->value =
                   !node->right_child->value;
                   break;
```

```
case and:     node->value =
              node->right_child->value &&
              node->left_child->value;
              break;
   case or:        node->value =
              node->right_child->value | |
              node->left_child->value;
              break;
    case true:    node->value = TRUE;
              break;
    case false:  node->value = FALSE;
   }
 }
}
```

Threaded Binary Trees

- Many null pointers in current representation of binary trees
  n: number of nodes; total links: 2n
  number of non-null links: n-1

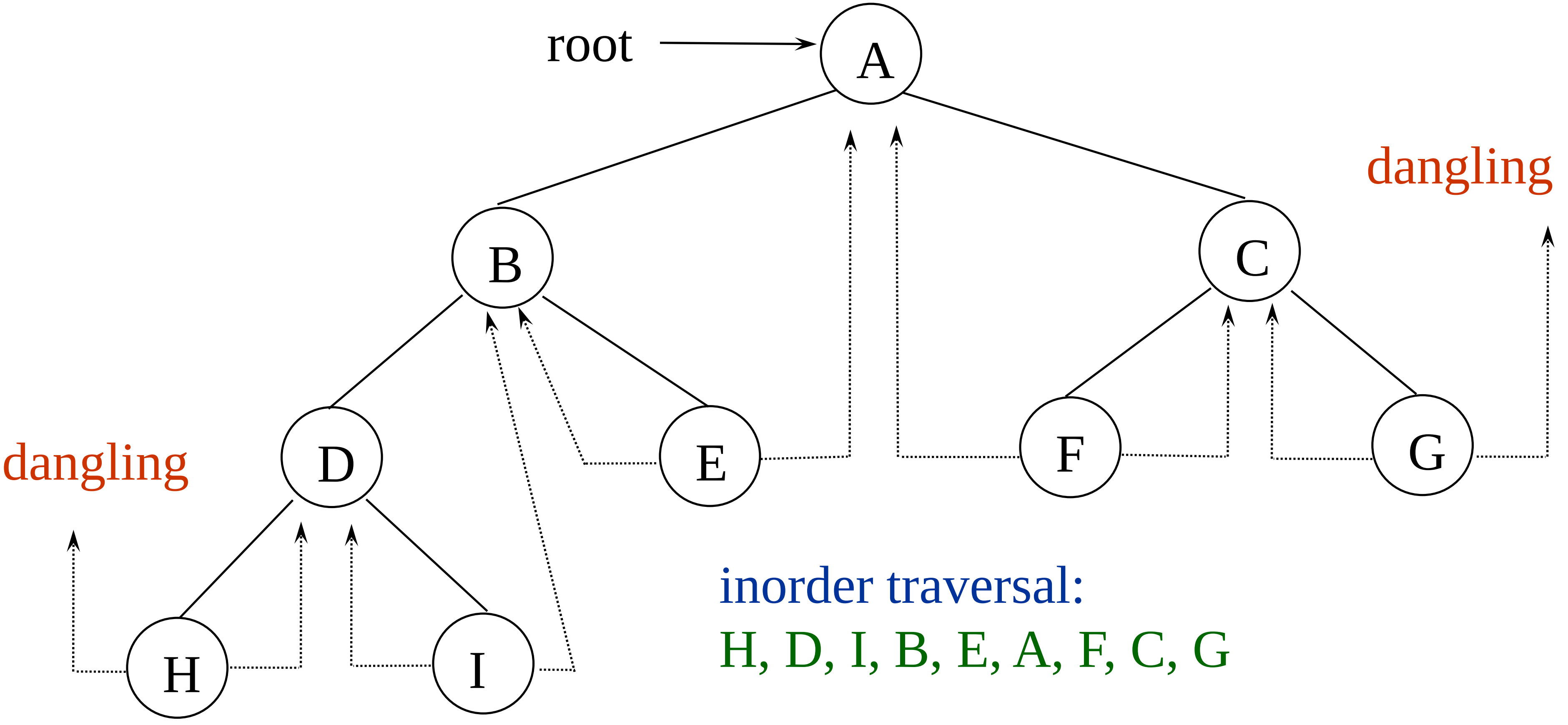
  null links: 2n-(n-1)=> n+1
- Replace these null pointers with some useful "threads".

Threaded Binary Trees (*Continued*)

If `ptr->left_child` is null,
replace it with a pointer to the node that would be visited *before* `ptr` in an *inorder traversal*

If `ptr->right_child` is null,
replace it with a pointer to the node that would be visited *after* `ptr` in an *inorder traversal*

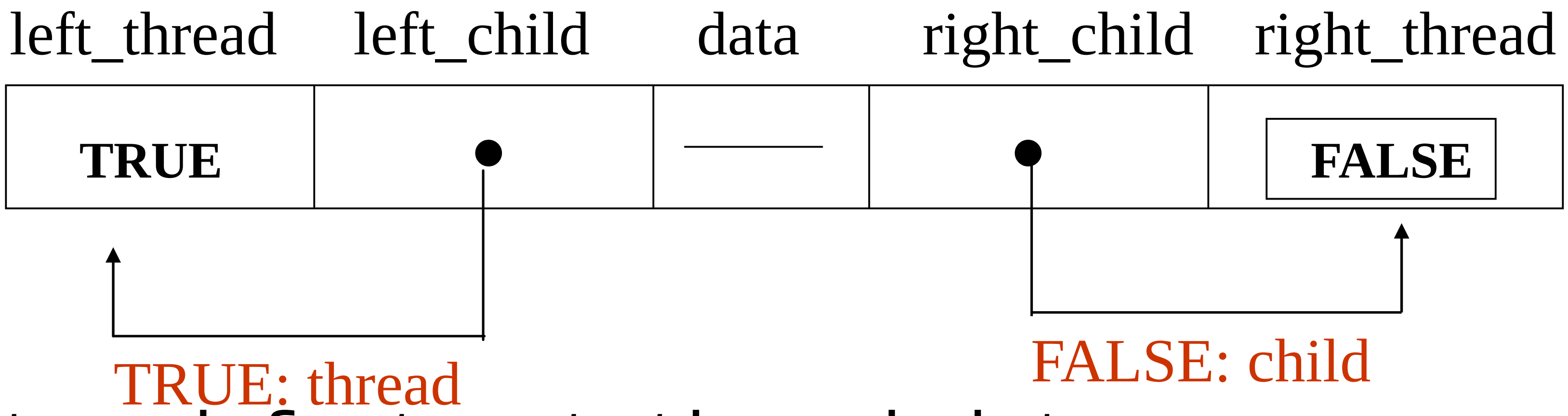
A Threaded Binary Tree



inorder traversal:

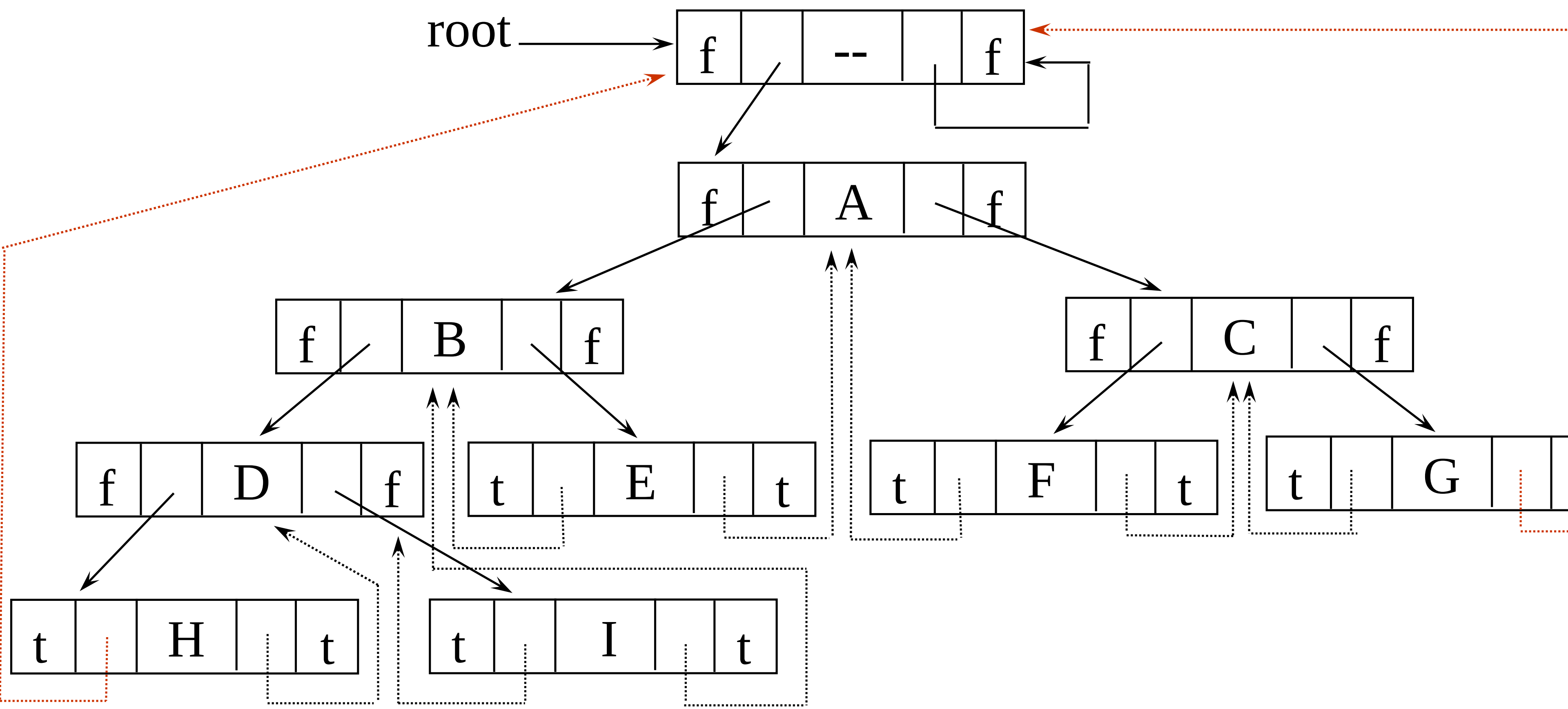
H, D, I, B, E, A, F, C, G

Data Structures for Threaded BT



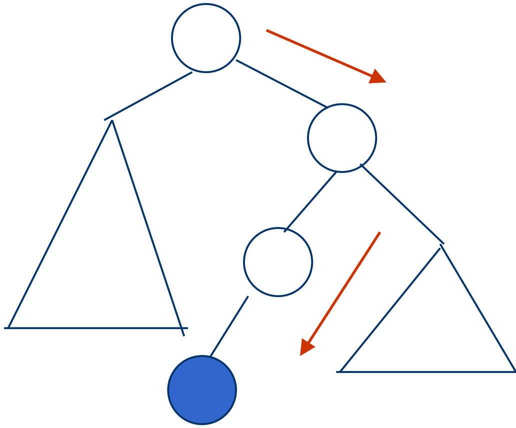

```
typedef struct threaded_tree
  *threaded_pointer;
typedef struct threaded_tree {
    short int left_thread;
    threaded_pointer left_child;
    char data;
    threaded_pointer right_child;
    short int right_thread;  };
```

Memory Representation of A Threaded BT



Next Node in Threaded BT

```
threaded_pointer insucc(threaded_pointer
  tree)
{
  threaded_pointer temp;
  temp = tree->right_child;
  if (!tree->right_thread)
    while (!temp->left_thread)
      temp = temp->left_child;
  return temp;
}
```



Inorder Traversal of Threaded BT

```
void tinorder(threaded_pointer tree)
{
/* traverse the threaded binary tree
  inorder */
    threaded_pointer temp = tree;
    for (;;) {
        temp = insucc(temp);
        if (temp==tree) break;
        printf("%3c", temp->data);
    }
}
```

O(n)

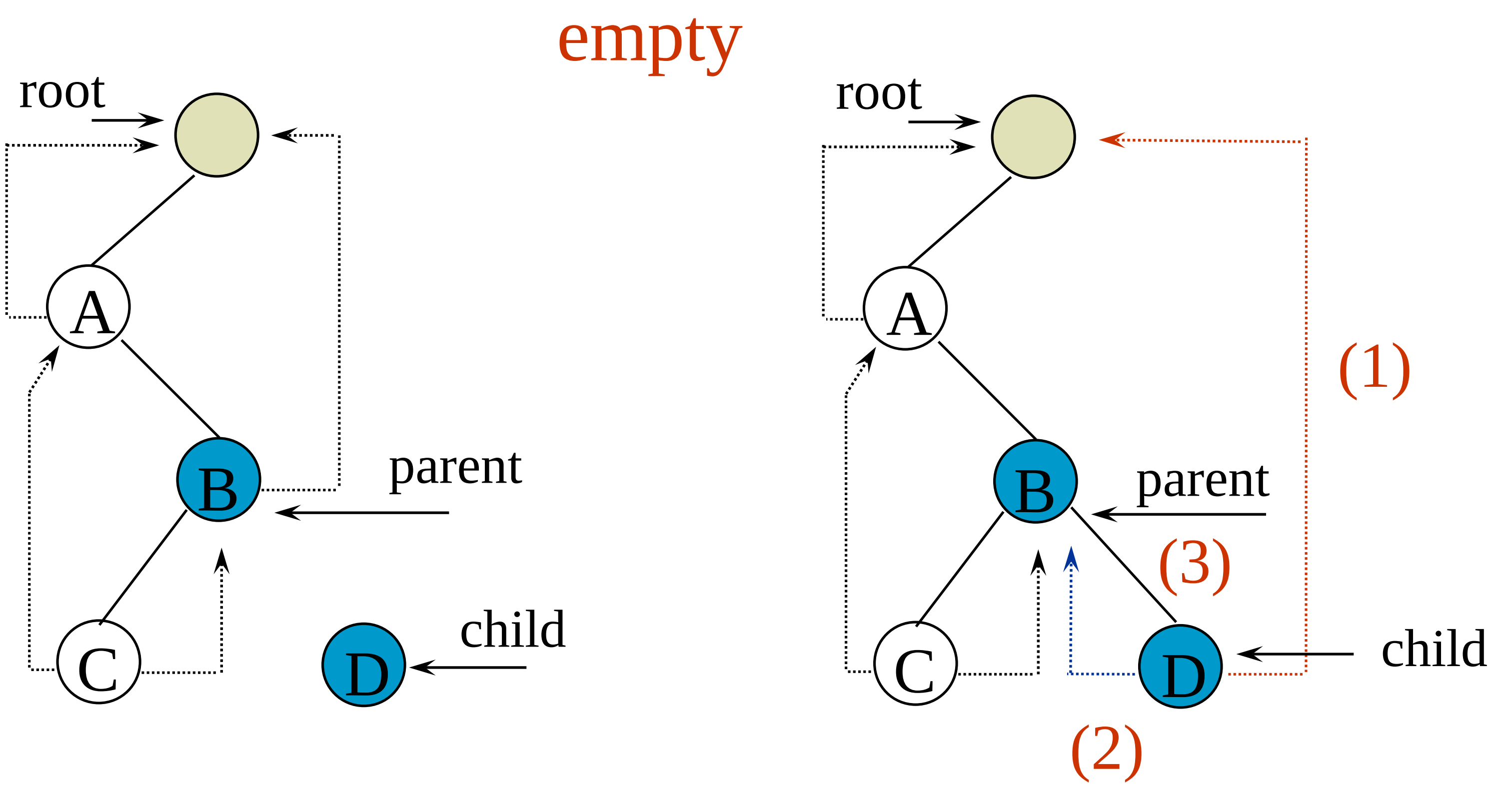
Inserting Nodes into Threaded BTs

- Insert `child` as the right child of node (`parent`)
  - change `parent->right_thread` to FALSE
  - set `child->left_thread` and `child->right_thread` to TRUE

1. set `child->right_child` to `parent->right_child`
2. set `child->left_child` to point to `parent`
3. change `parent->right_child` to point to `child`

Examples

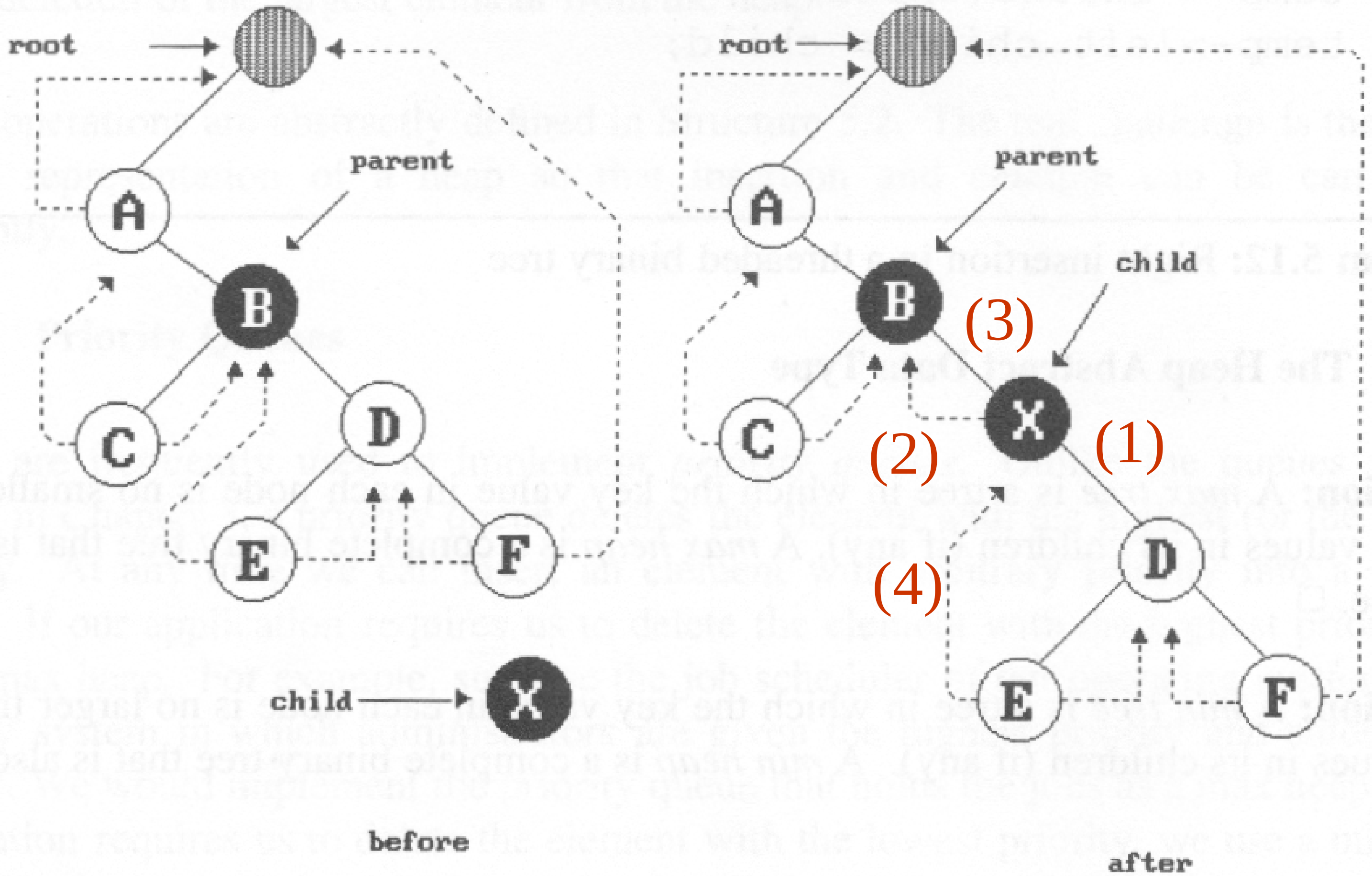
Insert a node D as a right child of B.



(a)

***Figure 5.24:** Insertion of child as a right child of parent in a threaded binary tree

nonempty



(b)

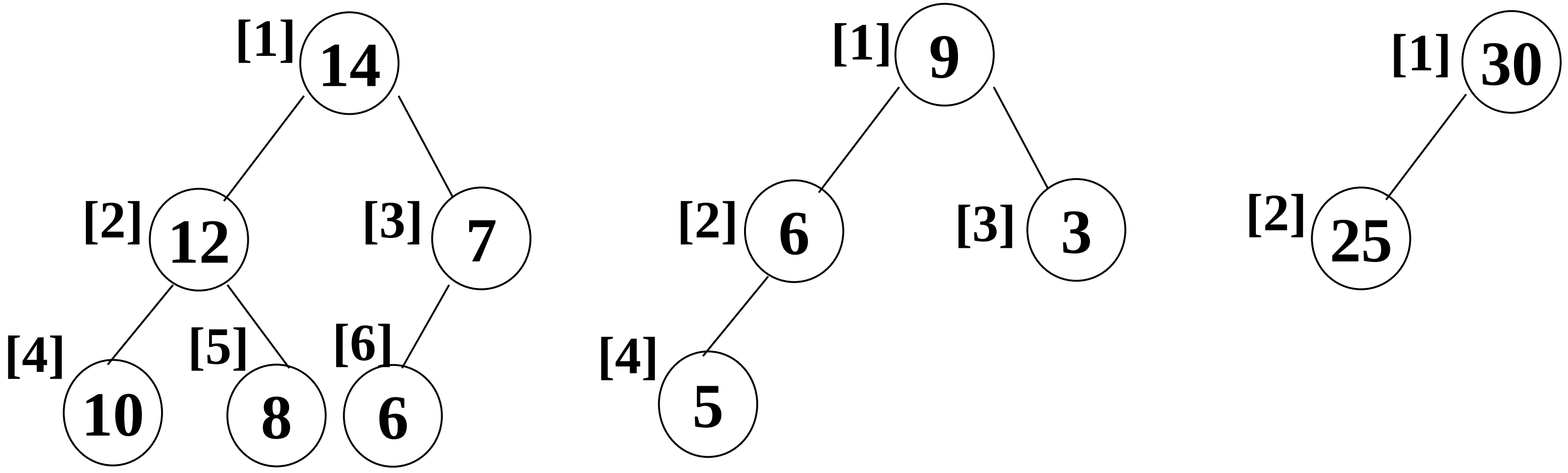
Right Insertion in Threaded BTs

```
void insertRight(threaded_pointer parent,
                 threaded_pointer child)
{
      threaded_pointer temp;
(1)   child->right_child = parent->right_child;
      child->right_thread = parent->right_thread;
(2)   child->left_child = parent;
      child->left_thread = TRUE;              case (a)
(3)   parent->right_child = child;
      parent->right_thread = FALSE;
      if (!child->right_thread) {             case (b)
        temp = insucc(child);
(4)     temp->left_child = child;
      }
}
```

Heap

- A *max tree* is a tree in which the key value in each node is *no smaller than* the key values in its children.
  - A *max heap* is a complete binary tree that is also a max tree.
- A *min tree* is a tree in which the key value in each node is *no larger than* the key values in its children.
  - A *min heap* is a complete binary tree that is also a min tree.
- Operations on heaps
  - creation of an empty heap
  - insertion of a new element into the heap
  - deletion of the largest element from the heap

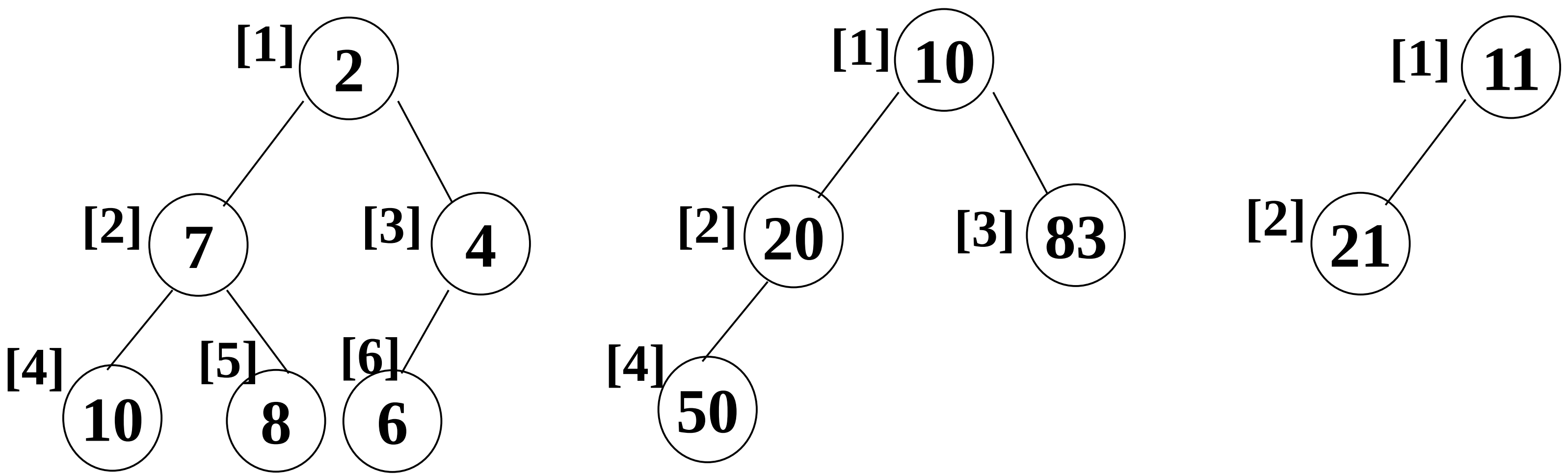
***Figure 5.25:** Max heaps



Property:

The root of max heap (min heap) contains the largest (smallest).

***Figure 5.26:** Min heaps



ADT for Max Heap

structure MaxHeap

- objects: a complete binary tree of n > 0 elements organized so that the value in each node is at least as large as those in its children

functions:

for all *heap* belong to *MaxHeap*, *item* belong to *Element*, *n*, *max_size* belong to integer

- MaxHeap Create(max_size)::= create an empty heap that can hold a maximum of max_size elements
- Boolean HeapFull(heap, n)::= if (n==max_size) return TRUE else return FALSE
- MaxHeap Insert(heap, item, n)::= if (!HeapFull(heap,n)) insert item into heap and return the resulting heap else return error
- Boolean HeapEmpty(heap, n)::= if (n>0) return FALSE else return TRUE
- Element Delete(heap,n)::= if (!HeapEmpty(heap,n)) return one instance of the **largest** element in the heap and remove it from the heap else return error

Application: priority queue

- Machine service (Example 5.1)
  - amount of time (min heap)
  - amount of payment (max heap)
- Factory (Example 5.2)
  - time tag

ADT MaxPriorityQuere是

物件：**n**個元素形成的集合**(n＞0)**，每個元素有一個鍵值

函式：對所有的**q∈MaxPriorityQueue**，**item∈Element**，**n**是整數

MaxPriorityQueue create(max_size)	::=	建立一個空的優先權佇列
Boolean isEmpty(q,n)	::=	**if(n＞0) return FALSE** **else return TRUE**
Element top(q,n)	::=	**if(!isEmpty(q,n)) return q**內最大的元素 **else return** 錯誤
Element pop(q,n)	::=	**if(!isEmpty(q,n)) return q**內最大的元素並把它從堆積中移除 **else return** 錯誤
MaxPriorityQueue push(q,item,n)	**::=**	把**item**插入**q**中並回傳優先權佇列的結果

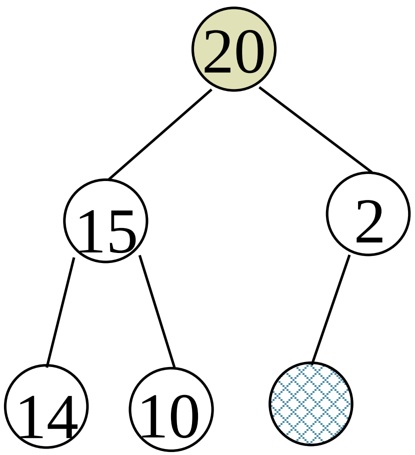
Data Structures

- unordered linked list
- unordered array
- sorted linked list
- sorted array
- heap

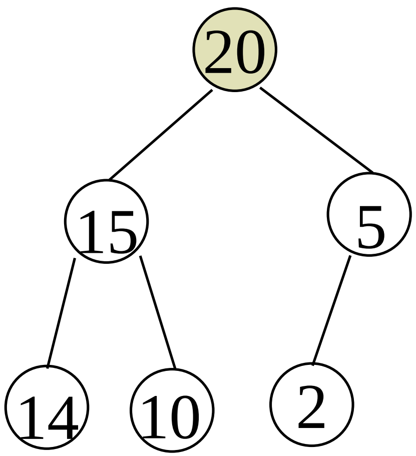
***Figure 5.27:** Priority queue representations*

Representation	Insertion	Deletion
Unordered array	$\Theta(1)$	$\Theta(n)$
Unordered linked list	$\Theta(1)$	$\Theta(n)$
Sorted array	O(n)	$\Theta(1)$
Sorted linked list	O(n)	$\Theta(1)$
Max heap	$O(\log_2 n)$	$O(\log_2 n)$

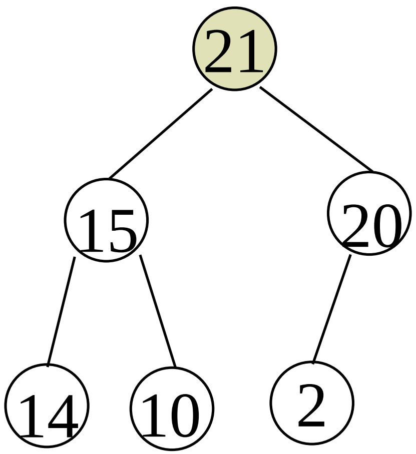
Example of Insertion to Max Heap



initial location of new node



insert 5 into heap



insert 21 into heap

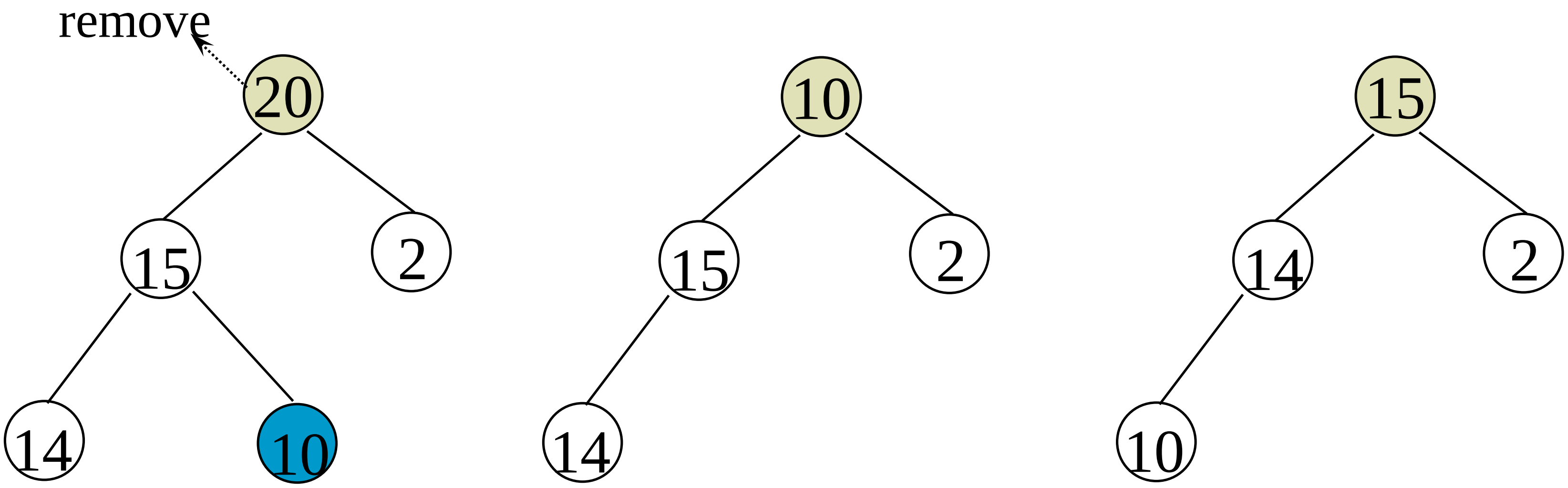
Insertion into a Max Heap

```
void push(element item, int *n)
{/* 把項目加入目前大小是n的最大堆積 */
  int i;
  if (HEAP_FULL(*n)) {
    fprintf(stderr, “the heap is full.\n”);
    exit(1);
  }
  i = ++(*n);
  while ((i!=1)&&(item.key>heap[i/2].key)) {
    heap[i] = heap[i/2]; // moving up to root
    i /= 2;
  }
  heap[i]= item;
}
```

$O(\log_2 n)$

$2^k-1=n \implies k=\lceil \log_2(n+1) \rceil$

Example of Deletion from Max Heap



Deletion from a Max Heap

```
element pop(int *n)
{/* 從堆積中刪除鍵最高的元素 */
  int parent, child;
  element item, temp;
  if (HEAP_EMPTY(*n)) {
    fprintf(stderr, "The heap is empty\n");
    exit(1);
  }
   /* save value of the element with the
   highest key */
  item = heap[1];
  /* use last element in heap to adjust heap */
  temp = heap[(*n)--];
  parent = 1;
  child = 2;
```

```
while (child <= *n) {
    /* find the larger child of the current
       parent */
    if ((child < *n)&&
        (heap[child].key<heap[child+1].key))
      child++;
    if (temp.key >= heap[child].key) break;
    /* move to the next lower level */
    heap[parent] = heap[child];
    child *= 2;
  }
  heap[parent] = temp;
  return item;
}
```

ADT Dictionary是

物件：**n**個資料對形成的集合**(n＞0)**，每個資料對有一個鍵值和搭配的項目

函式：

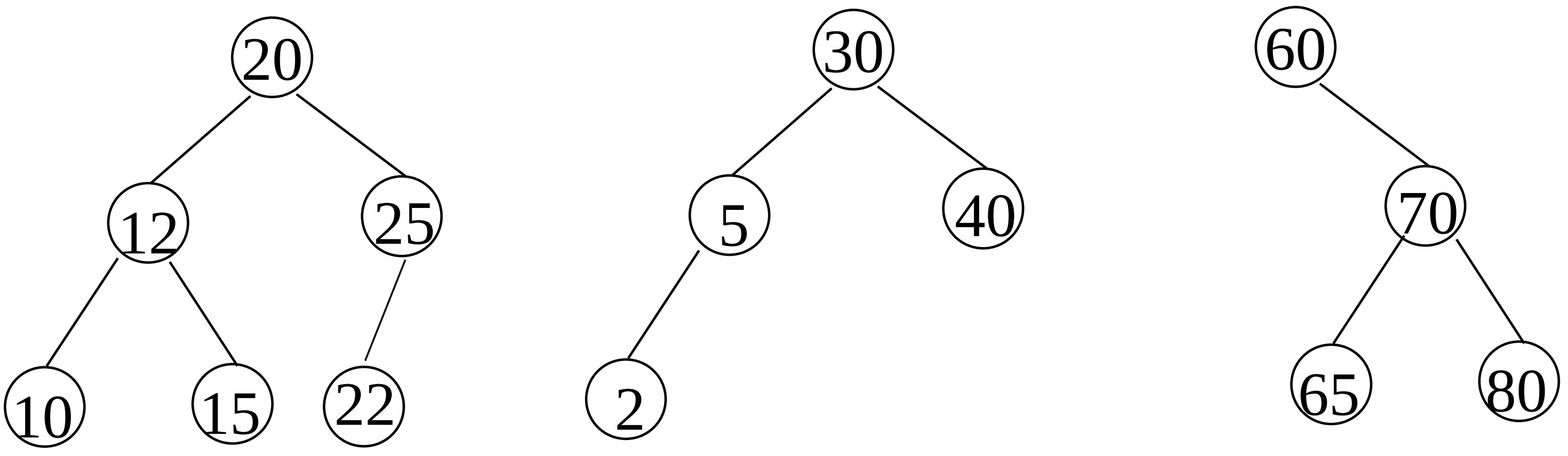
對於所有的**d∊Dictionary**，**item∊Item**，**k∊Key**，**n**是整數

Dictionary Create(max_size)	::=	建立一個空的字典
Boolean IsEmpty(d,n)	::=	**if(n>0) return FALSE** **else return TRUE**
Element Search(d,k)	::=	**return** 鍵值為**k**的項目 **return NULL** 如果沒有此元素
Element Delete(d,k)	::=	刪除並回傳**(**如果有**)**鍵值為**k**的項目
void Insert(d,item,k)	**::=**	把鍵值為**k**的**item**插入**d**中

Binary Search Tree

- Heap
  - a min (max) element is deleted. $O(\log_2 n)$
  - deletion of an arbitrary element $O(n)$
  - search for an arbitrary element $O(n)$
- Binary search tree
  - Every element has a unique key.
  - The keys in a nonempty left subtree (right subtree) are smaller (larger) than the key in the root of subtree.
  - The left and right subtrees are also binary search trees.

Examples of Binary Search Trees



Searching a Binary Search Tree

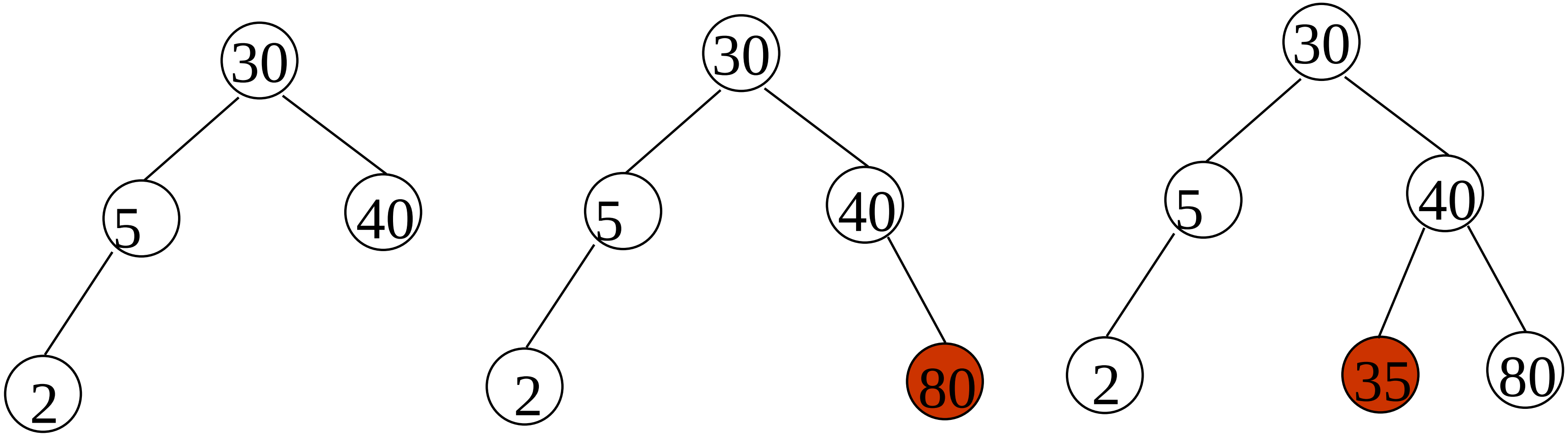
```
tree_pointer search(tree_pointer root,
                    int key)
{
 /* return a pointer to the node that
  contains key. If there is no such
  node, return NULL */

   if (!root) return NULL;
   if (key == root->data) return root;
   if (key < root->data)
       return search(root->left_child,
                     key);
   return search(root->right_child,key);
}
```

Another Searching Algorithm

```
tree_pointer iterSearch(tree_pointer
  tree, int key)
{
  while (tree) {
    if (key == tree->data) return tree;
    if (key < tree->data)
        tree = tree->left_child;
    else tree = tree->right_child;
  }
  return NULL;
}
```

O(h)

Insert Node in Binary Search Tree



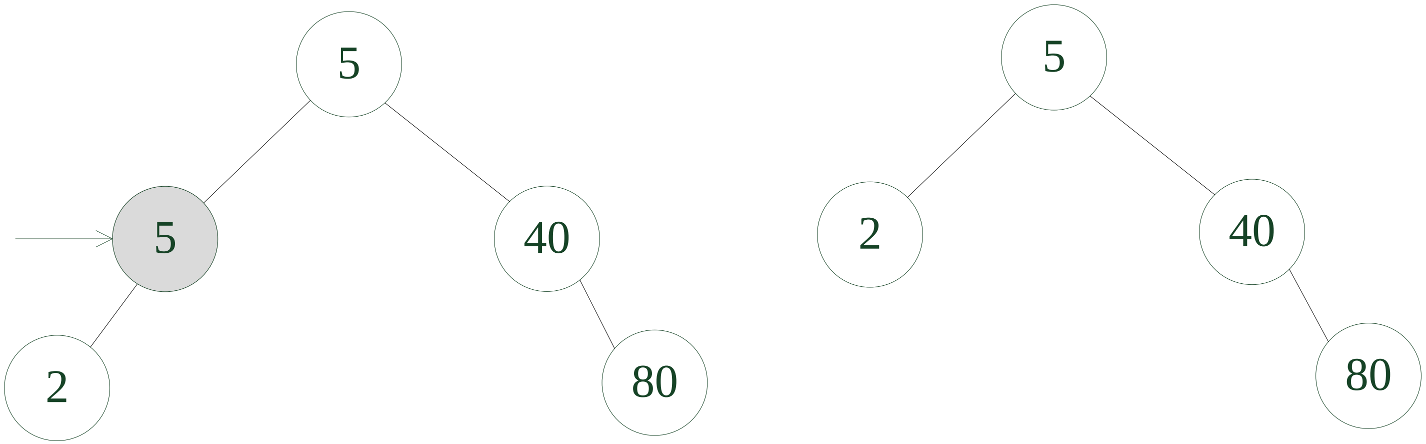
Insert 80

Insert 35

Insertion into a Binary Search Tree

```
void insert(tree_pointer *node, int k, iType
  theItem)
{tree_pointer ptr,
      temp = modified_search(*node, k);
  if (temp || !(*node)) {/* k不在樹中 */
   ptr = (tree_pointer) malloc(sizeof(node));
   if (IS_FULL(ptr)) {
     fprintf(stderr, "The memory is full\n");
     exit(1);
   }
   ptr->data.key = k; ptr->data.item = theItem;
   ptr->left_child = ptr->right_child = NULL;
   if (*node)
     if (k < temp->data) temp->left_child=ptr;
        else temp->right_child = ptr;
   else *node = ptr;
  }
}
```

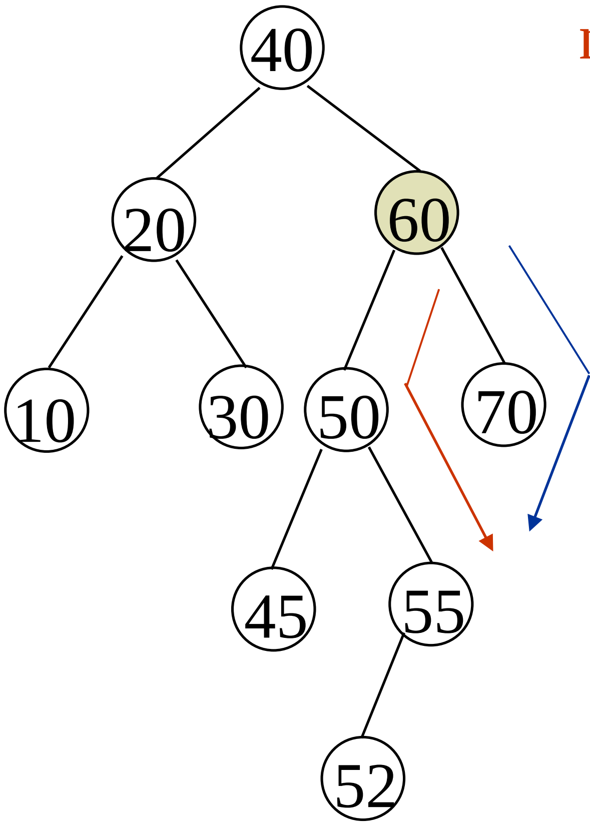
Deletion for a Binary Search Tree



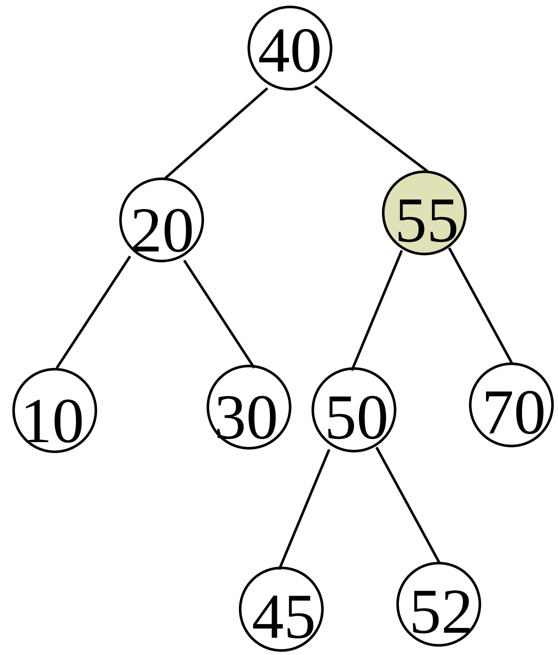
(a) (b)

Deletion for a Binary Search Tree

non-leaf node



Before deleting 60



After deleting 60

In the left, to find the maximum

In the right, to find the minmum

Split a Binary Search Tree

```
void split (nodePointer *theTree, int k, nodePointer *samll,
element *mid, nodePointer *big)
{ /* 根據鍵k來分割二元搜尋樹 */
    if (!theTree) {*small = *big = 0;  (*mid).key = -1; return;}
  /* 空樹 */
    nodePointer sHead, bHead, s, b, currentNode;
  /* 替small和big建立標頭節點 */
   MALLOC(sHead, sizeof(*sHead));
   MALLOC(bHead, sizeof(*bHead));
   s = sHead, b = bHead;
    /* 執行分割 */
    currentNode = *theTree;
    while (currentNode)
```

```
if (k < currentNode→data.key) { /* 加到big */
b→leftChild = currentNode; b = currentNode;
currentNode = currentNode→leftChild;   }
else if (k > currentNode→data.key) { /* 加到 small */
s→rightChild = currentNode; s = currentNode;
currentNode = currentNode→rightChild;  }

else { /* 在currentNode做分割 */
s→rightChild = currentNode→leftChild;
b→leftChild = currentNode→rightChild;
*small = sHead→rightChild; free(sHead);
*big = bHead→leftChild; free(bHead);
(*mid).item = currentNode→data.item;
(*mid).key = currentNode→data.key;
free(currentNode);
return; }
```

```
/* 沒有鍵為k的字典對 */
s→rightChild = b→leftChild = 0;
*small = sHead→rightChild; free(sHead);
*big = bHead→leftChild; free(bHead);
 (*mid).key = -1;
return;
}
```

Selection Trees

(1) Winner tree
(2) Loser tree

Sequential allocation scheme
(complete binary tree)

Winner tree

Each node represents the smaller of its two children.

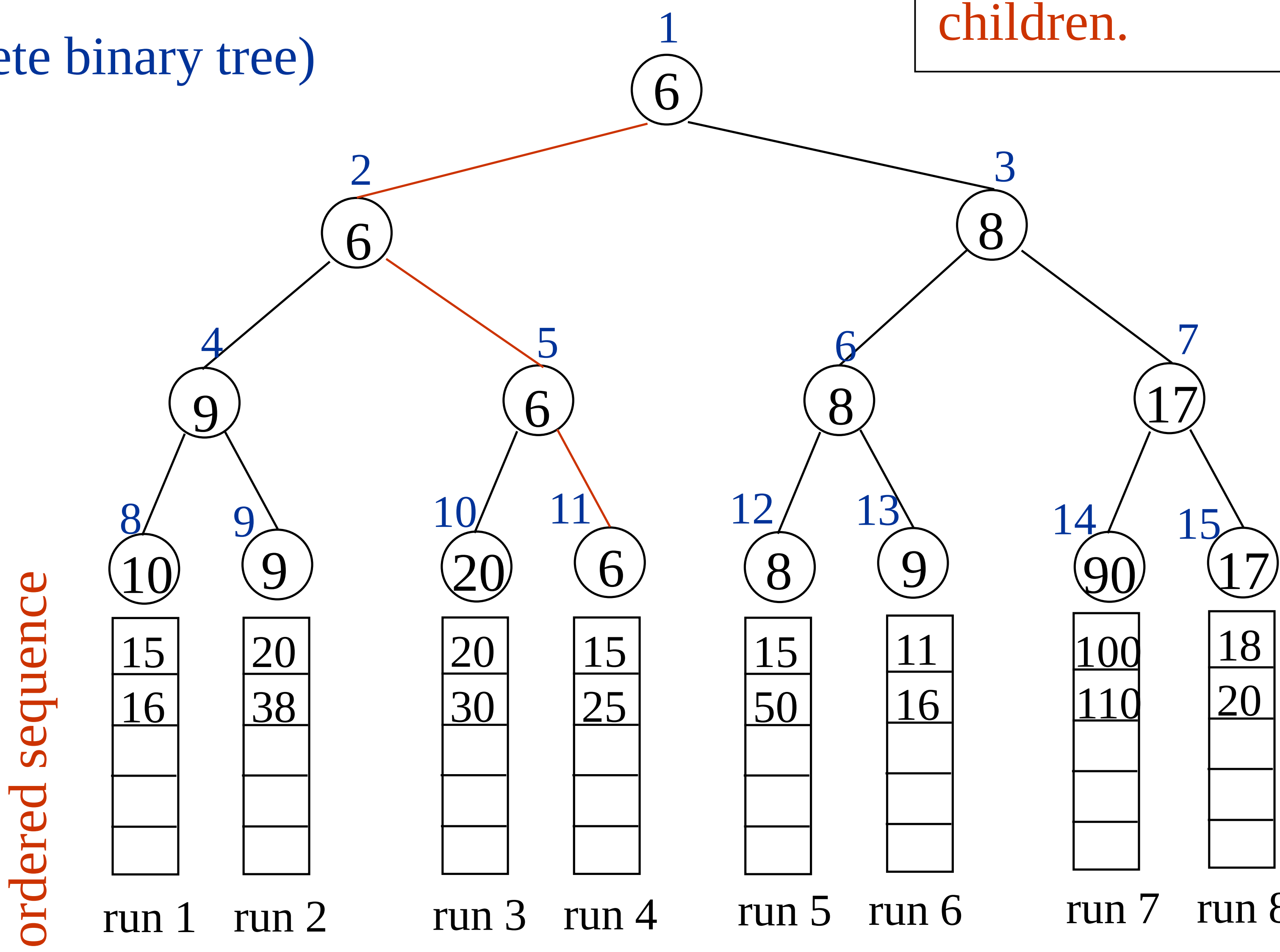
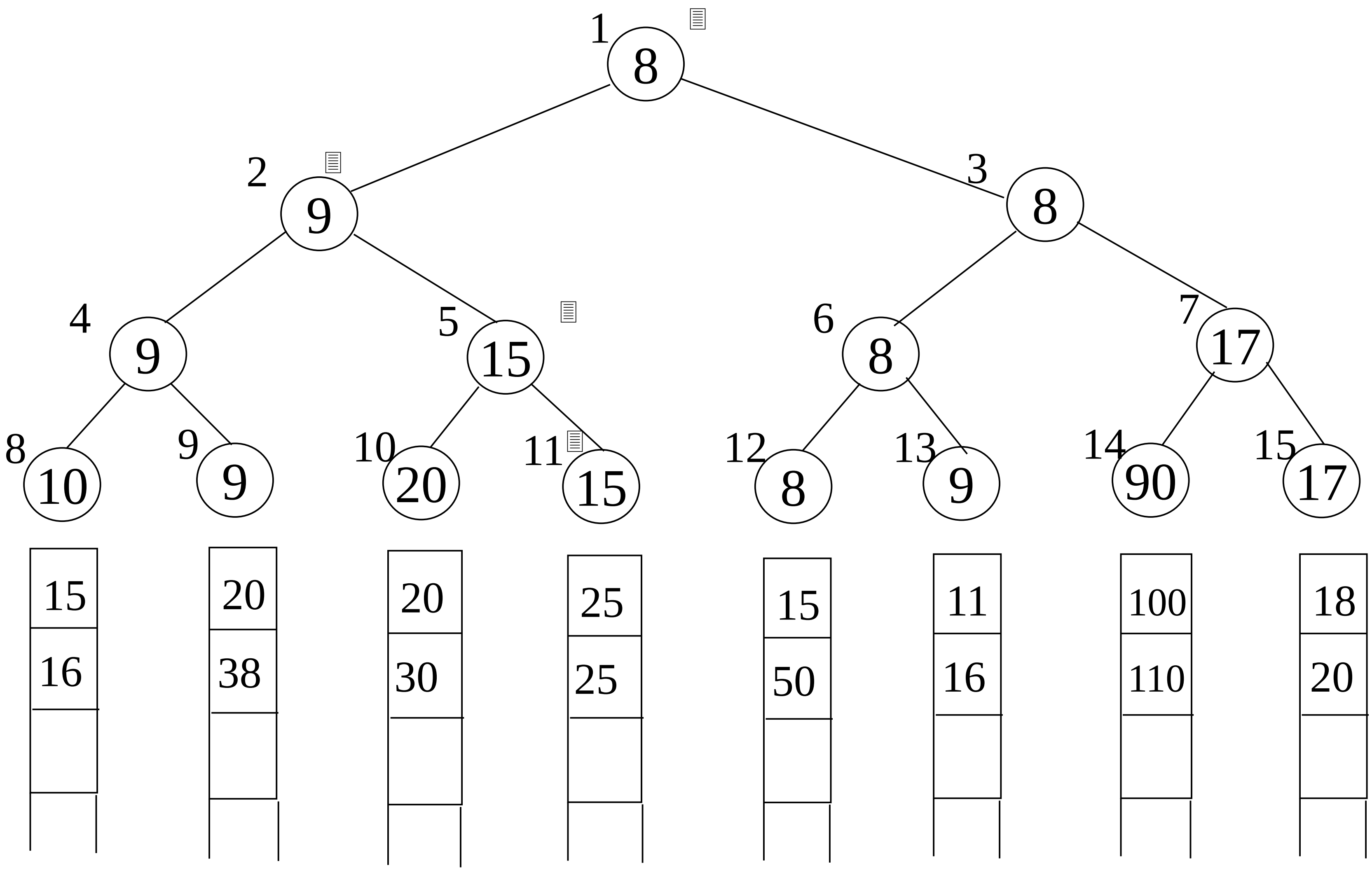


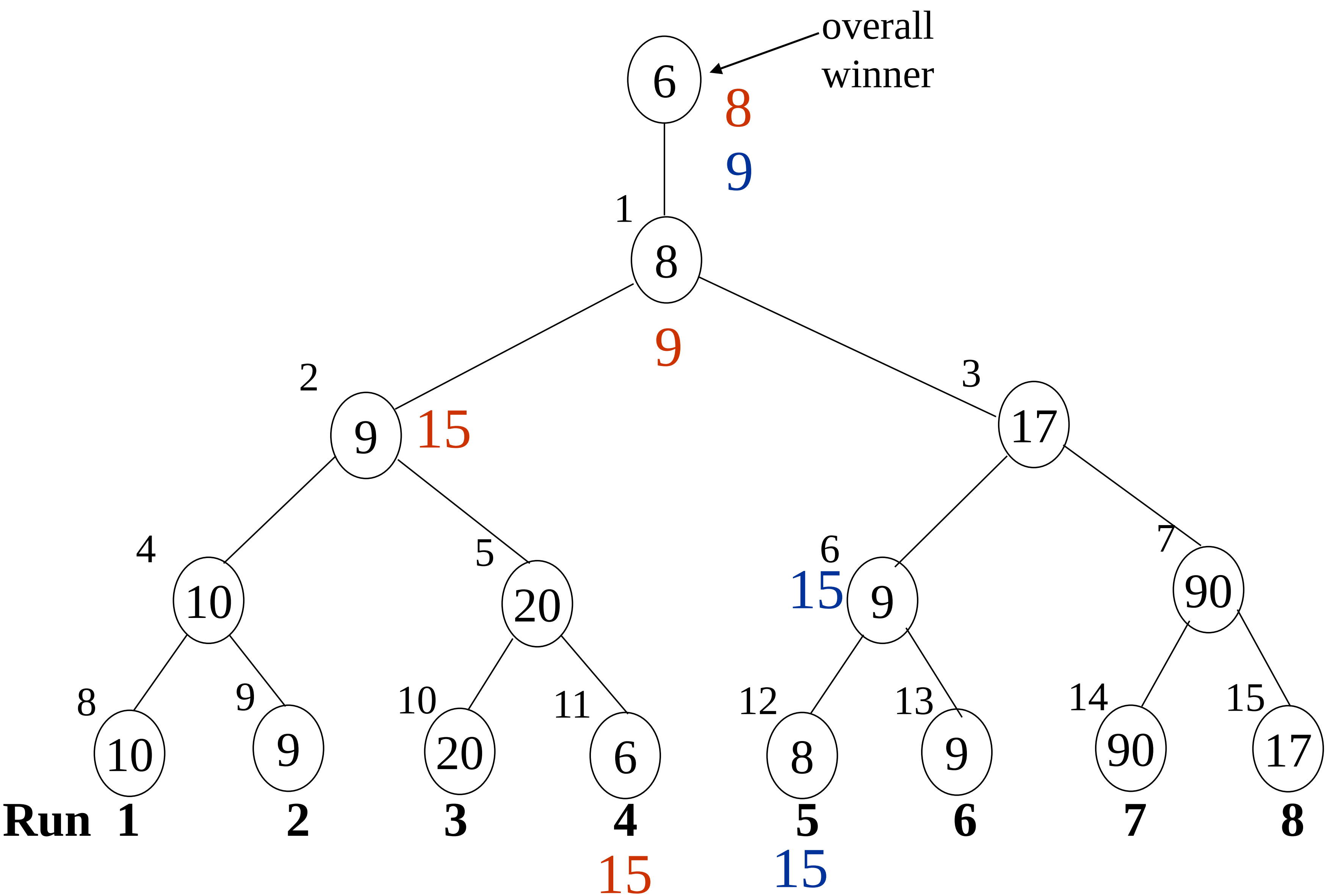
***Figure 5.35:** Selection tree of Figure 5.34 after one record has been output and the tree restructured (nodes that were changed are ticked)



Analysis

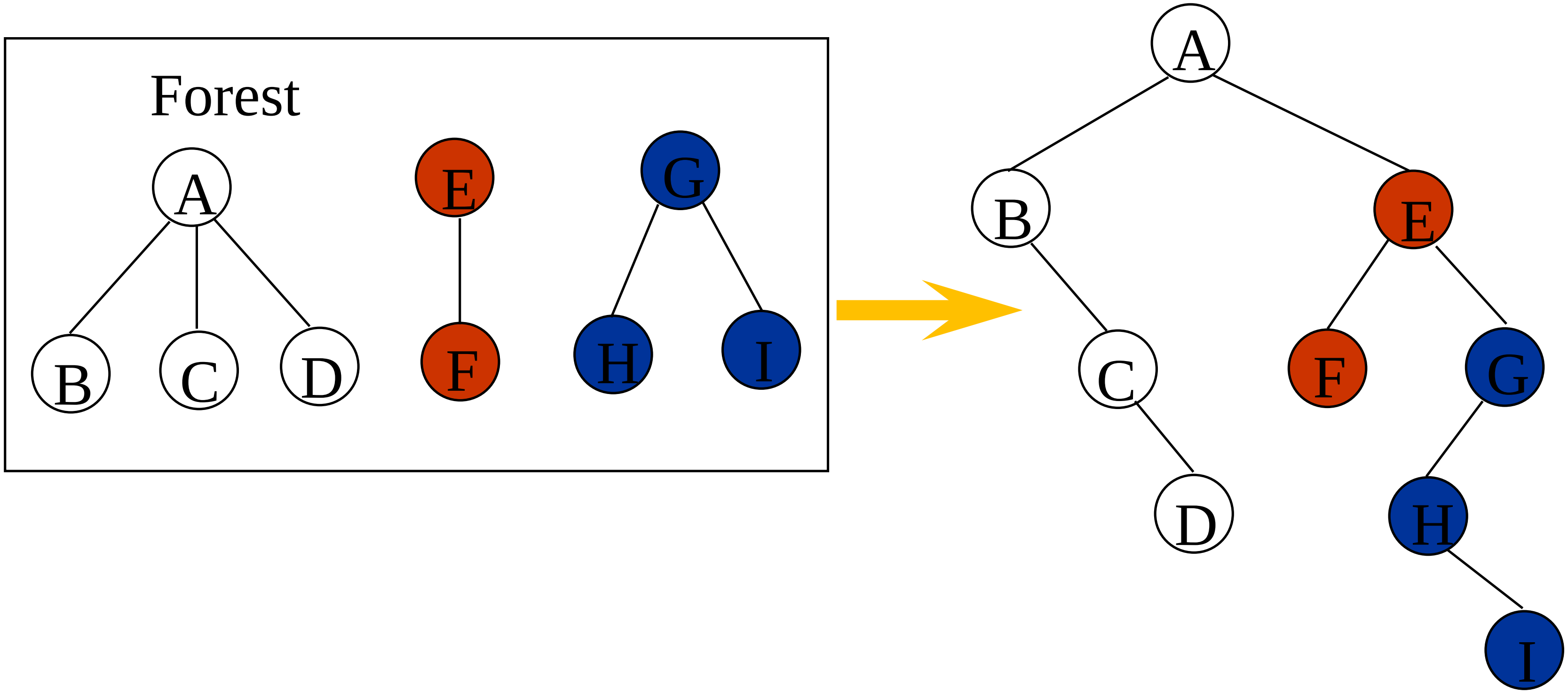
- K: # of runs
- n: # of records
- setup time: O(K) (K-1)
- restructure time: $O(\log_2 K)$ $\lceil \log_2(K+1) \rceil$
- merge time: $O(n\log_2 K)$
- slight modification: loser tree
  - consider the parent node only (vs. sibling nodes)

***Figure 5.34:** Tree of losers corresponding to Figure 5.32



Forest

- Definition: **A forest is a set of n >= 0 disjoint trees**



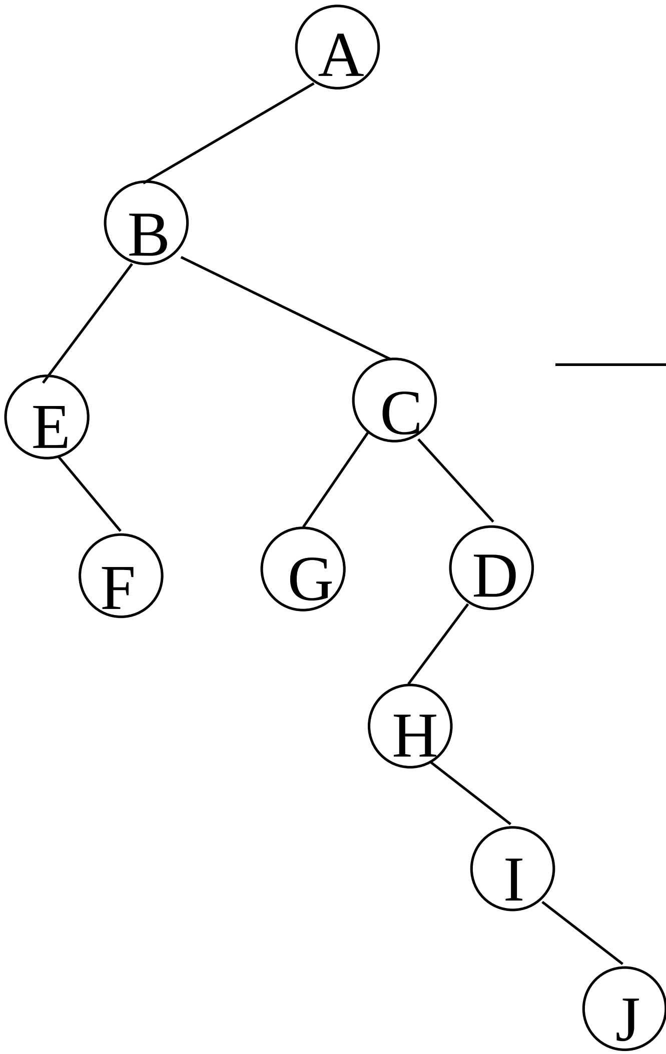
Transform a forest into a binary tree

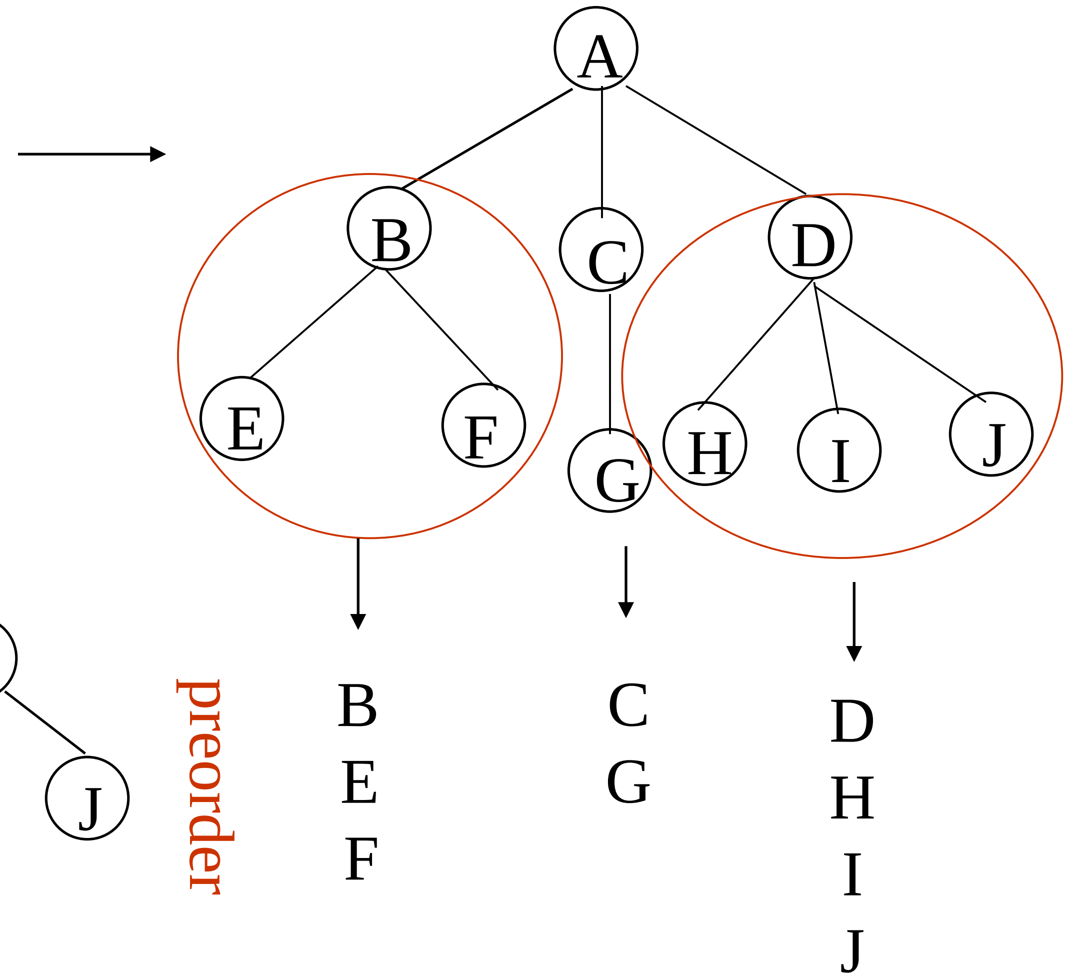
- T1, T2, …, T*n*: a forest of trees
  B(T1, T2, …, Tn): a binary tree corresponding to this forest
- Algorithm
  (1) empty, if n = 0
  (2) has root equal to root(T1)
  has left subtree equal to B(T11,T12,…,T1*m*)
  has right subtree equal to B(T2,T3,…,Tn)

Forest Traversals

- Preorder (V)
  - If F is empty, then return
  - Visit the root of the first tree of F
  - Taverse the subtrees of the first tree in tree preorder
  - Traverse the remaining trees of F in preorder
- Inorder (LVR)
  - If F is empty, then return
  - Traverse the subtrees of the first tree in tree inorder
  - Visit the root of the first tree
  - Traverse the remaining trees of F is indorer

inorder: EFBGCHIJDA
preorder: ABEFCGDHIJ

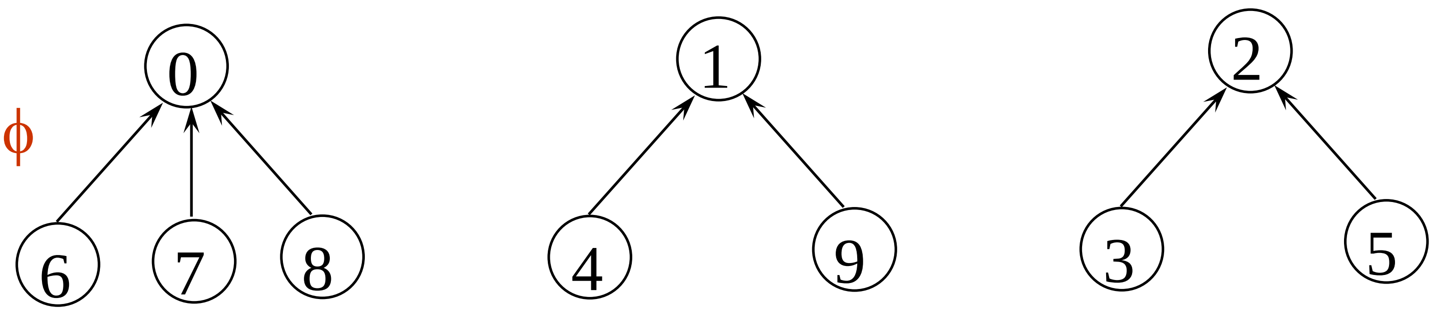




Set Representation

- $S_1=\{0, 6, 7, 8\}$, $S_2=\{1, 4, 9\}$, $S_3=\{2, 3, 5\}$

$S_i \cap S_j = \phi$

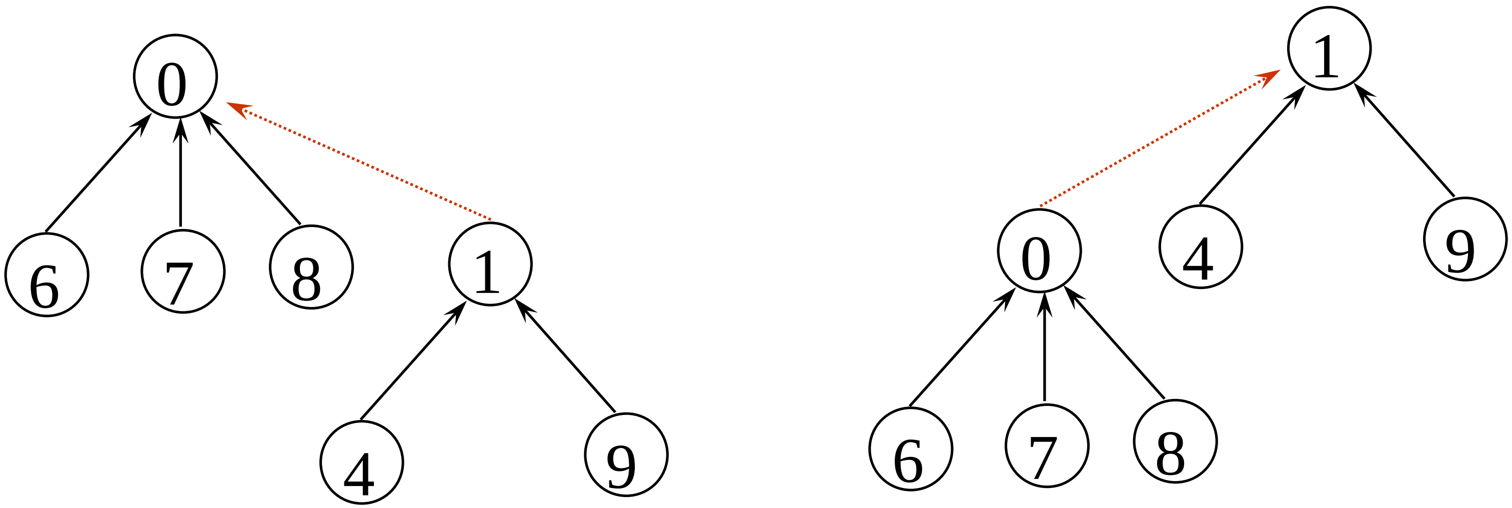


- Two operations considered here
  - *Disjoint set union* $S_1 \cup S_2=\{0,6,7,8,1,4,9\}$
  - *Find*(*i*): Find the set containing the element *i*.

    $3 \in S_3$, $8 \in S_1$

Disjoint Set Union

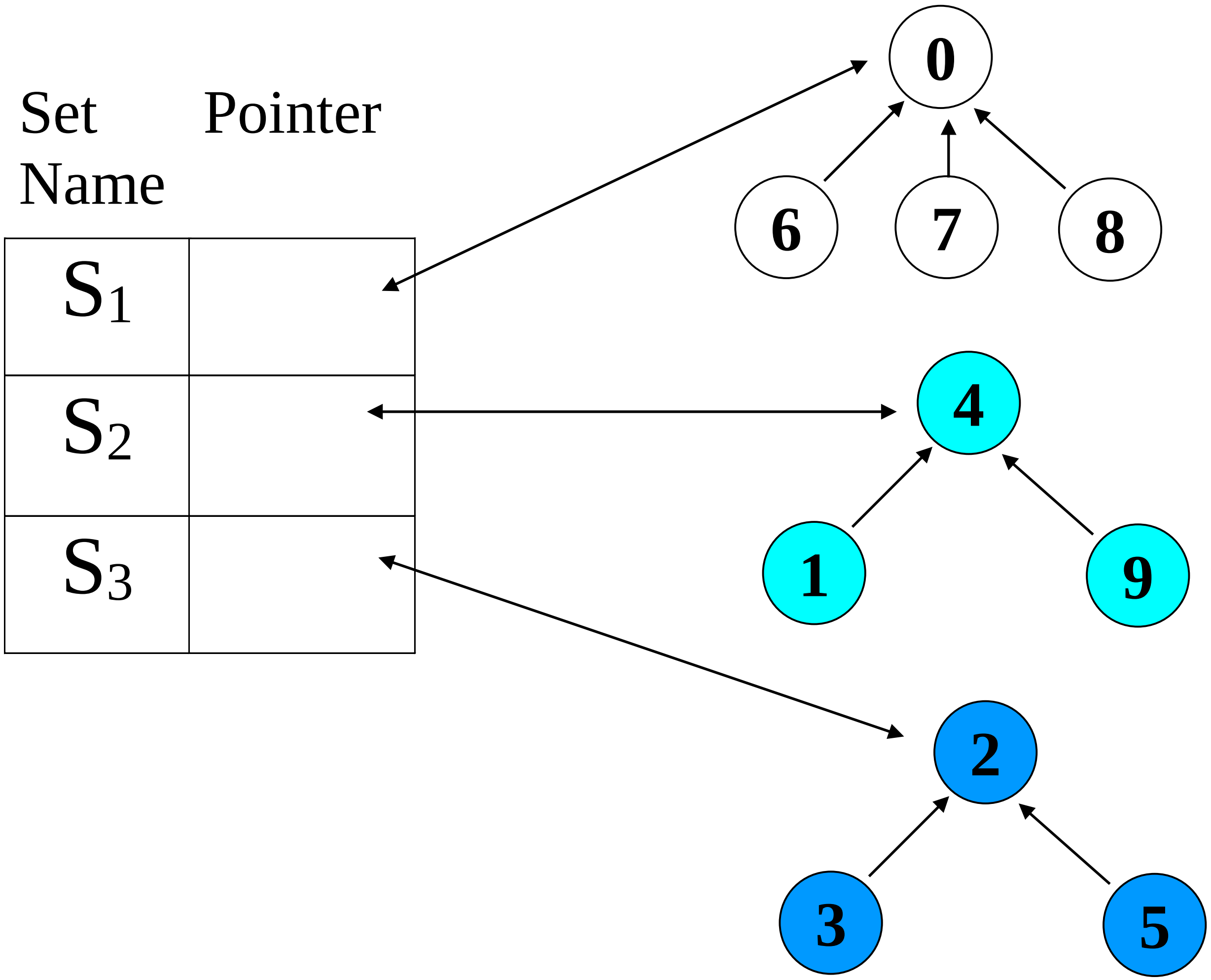
Make one of the trees a subtree of the other



Possible representation for S_1 *union* S_2

$S_1 \cup S_2$

***Figure 5.39:**Data Representation of $S_1 S_2$ and S_3



Array Representation for Set

i	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
parent	-1	4	-1	2	-1	2	0	0	0	4

```
int simpleFind(int i)
{
    for (; parent[i]>=0; i=parent[i]);
    return i;
}

void simpleUnion(int i, int j)
{
    parent[i]= j;
}
```

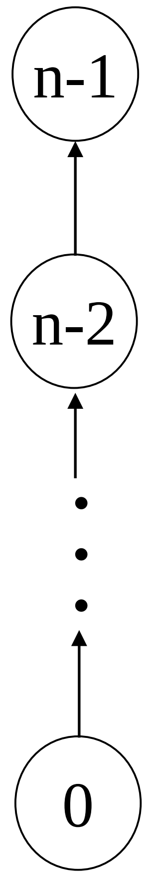
***Figure 5.41:** Degenerate tree (退化樹)

union operation

O(n) n-1

find operation

O(n^2) $\sum_{i=2}^{n} i$



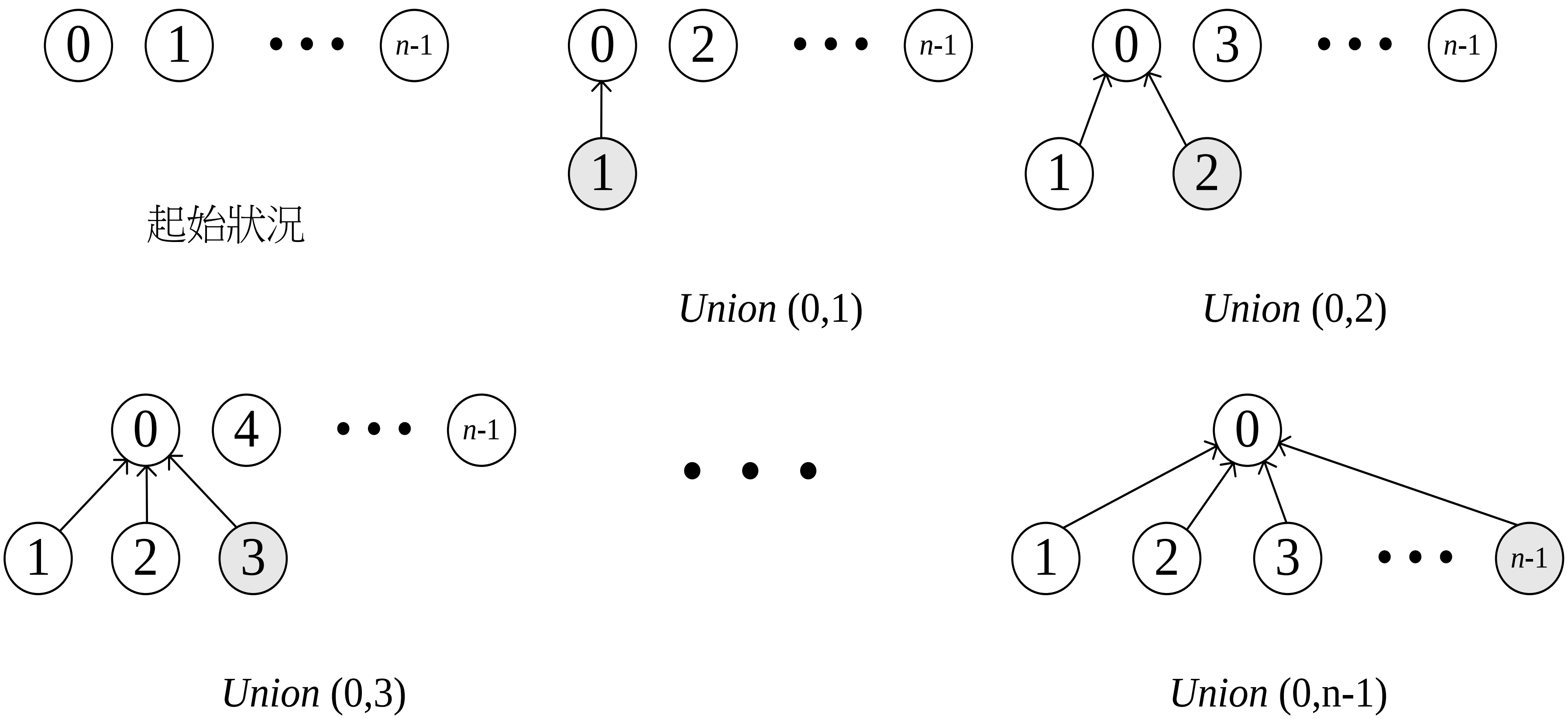
degenerate tree

union(0,1), find(0)
union(1,2), find(0)
.
.
.
union(n-2,n-1),find(0)

***Figure 5.42:** Trees obtained using the weighting rule

weighting rule for union(i,j):

if # of nodes in i < # in j then make j the parent of i

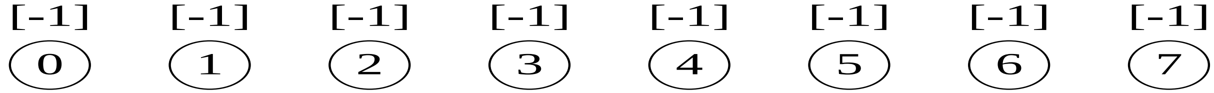


Modified Union Operation

```
void weightedUnion(int i, int j)
{
    //parent[i]=-count[i] and parent=-count[j]
    int temp = parent[i]+ parent[j];
    if (parent[i]>parent[j]) {
        parent[i]=j;
/* make j the new root*/
        parent[j]=temp;
    }
    else {
        parent[j]=i;
/* make i the new root*/
        parent[i]=temp;
    }
}
```

Keep a count in the root of tree

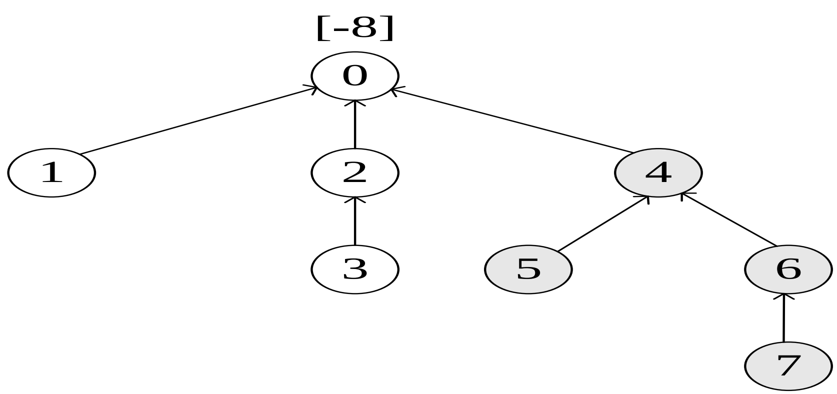
If the number of nodes in tree i is less than the number in tree j, then make j the parent of i; otherwise make i the parent of j.



(a) 一開始樹的高度都是1

(b) 執行*Union* (0,1)，(2,3)，(4,5)，與 (6,7) 後樹之高度為 2

(c) 執行*Union* (0,2) 與 (4,6) 後樹之高度為 3



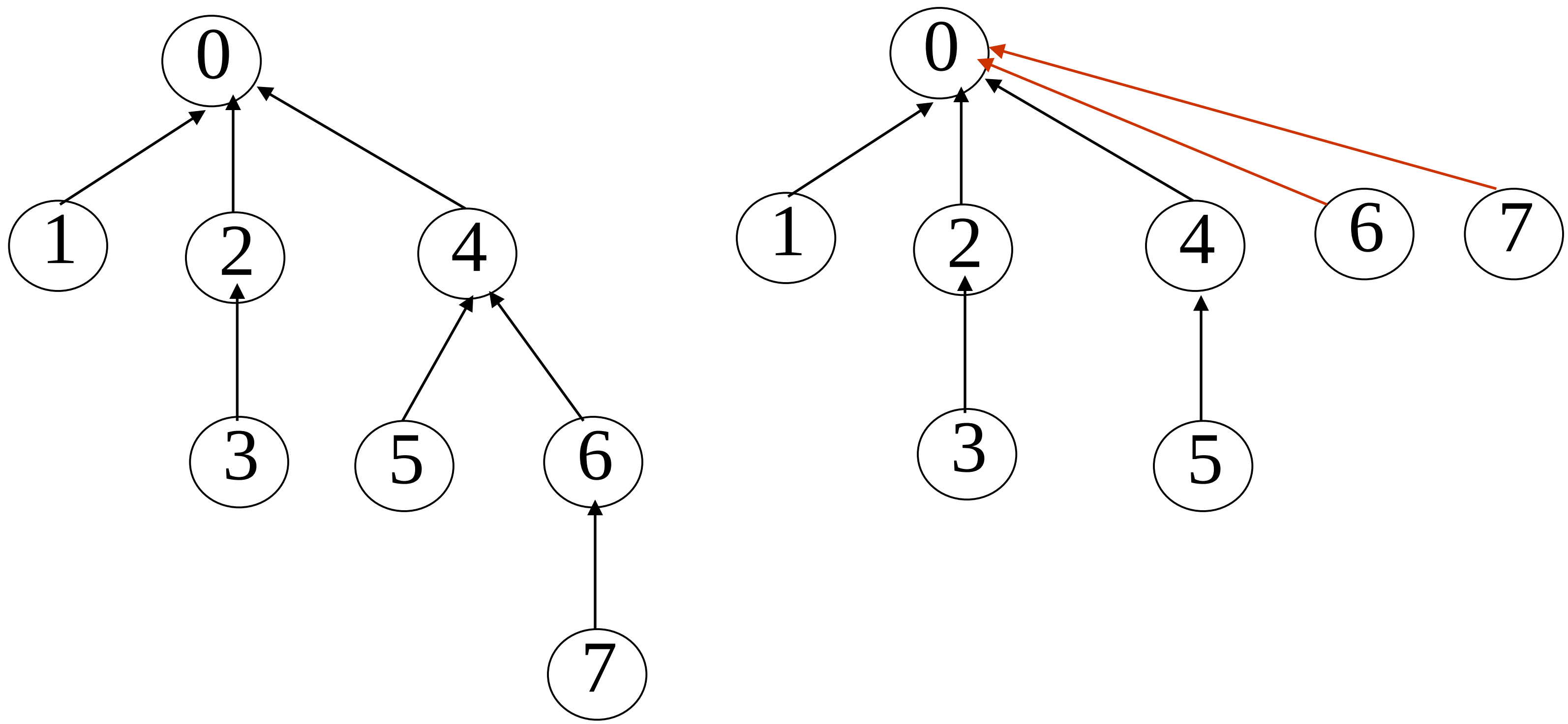
(d) 執行*Union* (0,4) 後樹之高度為 4

Figure 5.43: Trees ach

collapsingFind(i) Operation

```
int collapsingFind(int i)
{
    int root, trail, lead;
    for (root=i; parent[root]>=0;
                      root=parent[root]);
    for (trail=i; trail!=root;
                      trail=lead) {
        lead = parent[trail];
        parent[trail]= root;
    }
    return root:
}
```

If j is a node on the path from i to its root then make j a child of the root



	find(7)	find(7)	find(7)	find(7)	find(7)	find(7)	find(7)	find(7)
go up	3	1	1	1	1	1	1	1
reset	2							

13 moves (vs. 24 moves)

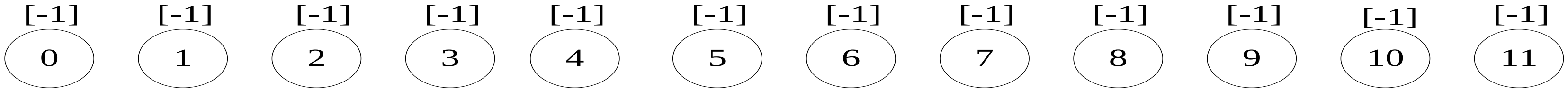
Application to Equivalence Classes

- Find equivalence class $i \equiv j$
- Find S_i and S_j such that $i \in S_i$ and $j \in S_j$ (two finds)
  - $S_i = S_j$ do nothing
  - $S_i \neq S_j$ union(S_i , S_j)
- example

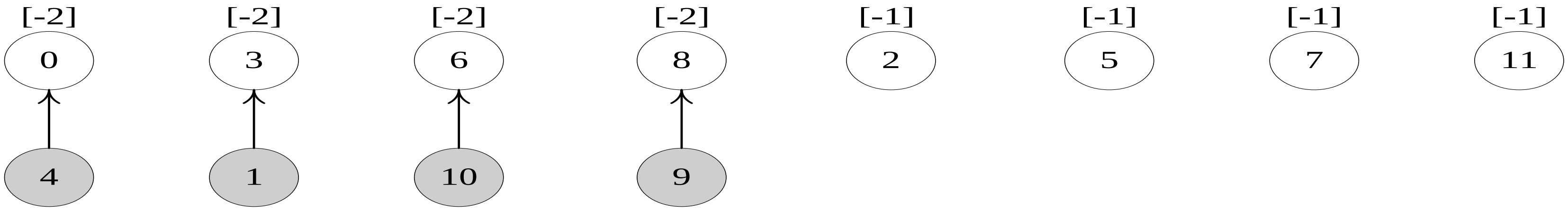
  $0 \equiv 4,\ 3 \equiv 1,\ 6 \equiv 10,\ 8 \equiv 9,\ 7 \equiv 4,\ 6 \equiv 8,$

  $3 \equiv 5,\ 2 \equiv 11,\ 11 \equiv 0$

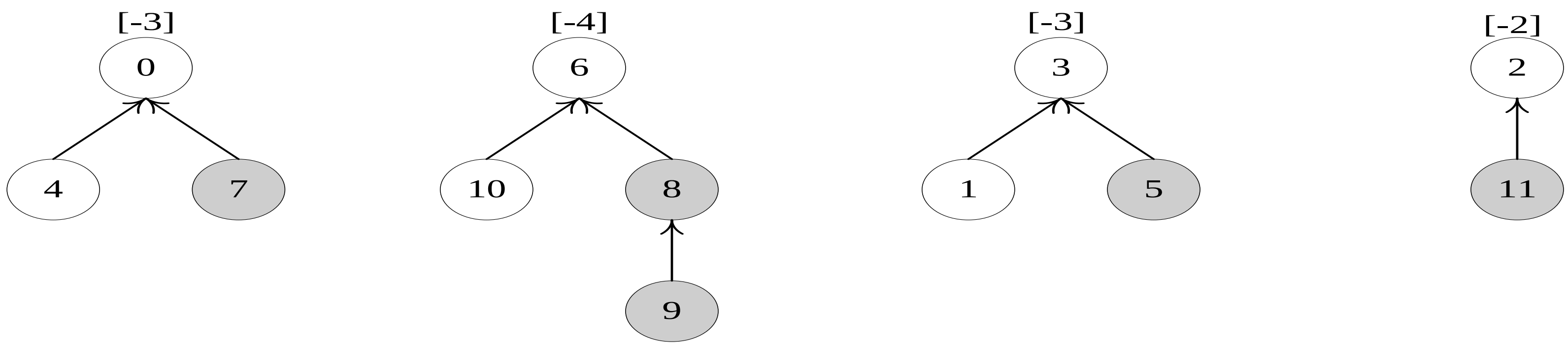
  $\{0, 2, 4, 7, 11\}, \{1, 3, 5\}, \{6, 8, 9, 10\}$



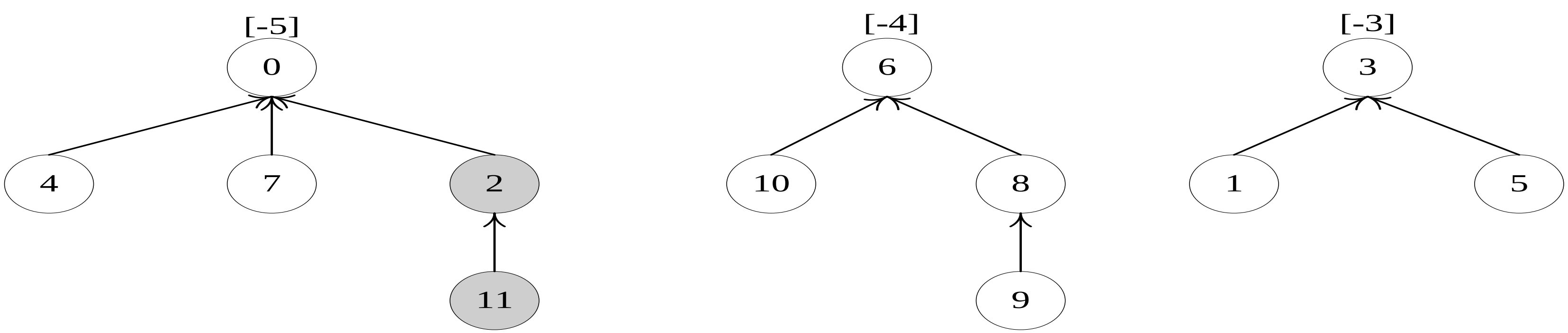
(a) 起始樹



(b) 處理完 0 ≡ 4, 3 ≡ 1, 6 ≡ 10, 8 ≡ 9 後高度為 2 的樹



(c) 處理完 7 ≡ 4, 6 ≡ 8, 3 ≡ 5, 2 ≡ 11 後的樹



(d) 處理完 11 ≡ 0 後的樹

preorder: A B C D E F G H I

inorder: B C A E D G H F I