CHAPTER 6

GRAPHS

All the programs in this file are selected from

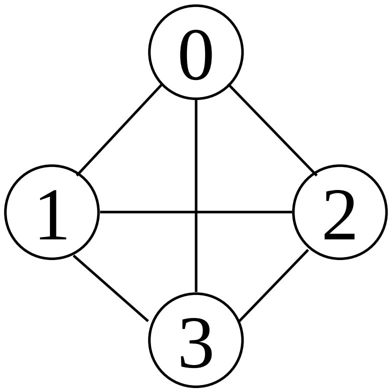
Ellis Horowitz, Sartaj Sahni, and Susan Anderson-Freed
“Fundamentals of Data Structures in C”,

Definition

- A graph G consists of two sets
  - – a finite, nonempty set of vertices V(G)
  - – a finite, possible empty set of edges E(G)
  - – G(V, E) represents a graph
- An undirected graph is one in which the pair of vertices in a edge is unordered, $(v_0, v_1) = (v_1,v_0)$
- A directed graph is one in which each edge is a directed pair of vertices, $<v_0, v_1> \,!= <v_1,v_0>$

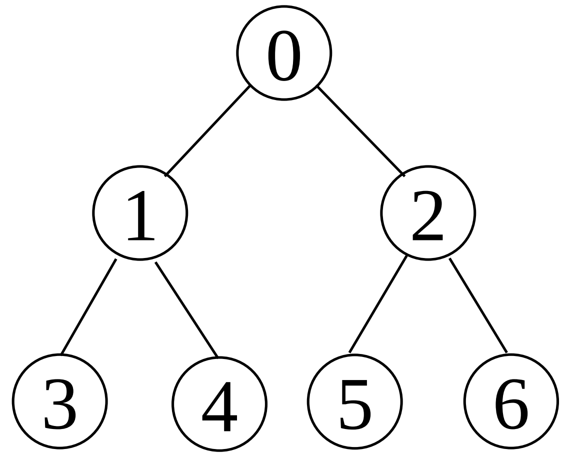
tail head

Examples for Graph



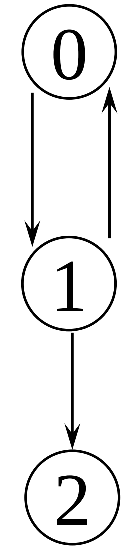
G_1

complete graph



G_2

incomplete graph



G_3

$V(G_1)=\{0,1,2,3\}$ $E(G_1)=\{(0,1),(0,2),(0,3),(1,2),(1,3),(2,3)\}$

$V(G_2)=\{0,1,2,3,4,5,6\}$ $E(G_2)=\{(0,1),(0,2),(1,3),(1,4),(2,5),(2,6)\}$

$V(G_3)=\{0,1,2\}$ $E(G_3)=\{<0,1>,<1,0>,<1,2>\}$

complete undirected graph: n(n-1)/2 edges

complete directed graph: n(n-1) edges

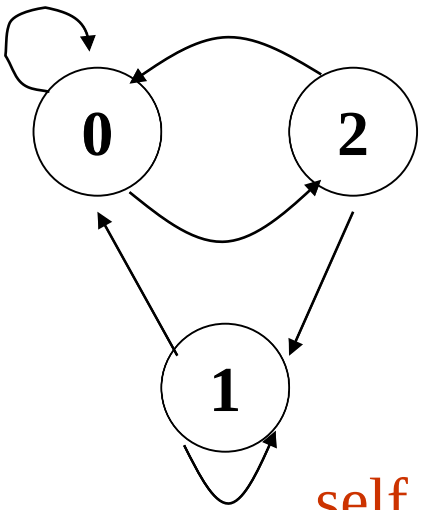
Complete Graph

- A complete graph is a graph that <u>has the maximum number of edges</u>
  - for undirected graph with n vertices, the maximum number of edges is $n(n-1)/2$
  - for directed graph with n vertices, the maximum number of edges is $n(n-1)$
  - example: G1 is a complete graph

Adjacent and Incident

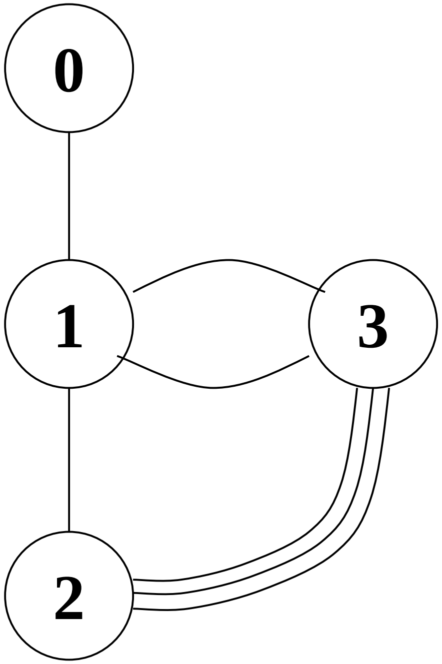
- If (v_0, v_1) is an edge in an undirected graph,
  - v_0 and v_1 are adjacent
  - The edge (v_0, v_1) is incident on vertices v_0 and v_1
- If $<v_0, v_1>$ is an edge in a directed graph
  - v_0 is adjacent **to** v_1, and v_1 is adjacent **from** v_0
  - The edge $<v_0, v_1>$ is incident on v_0 and v_1

***Figure 6.3:**Example of a graph with feedback loops and a multigraph



self edge

(a)



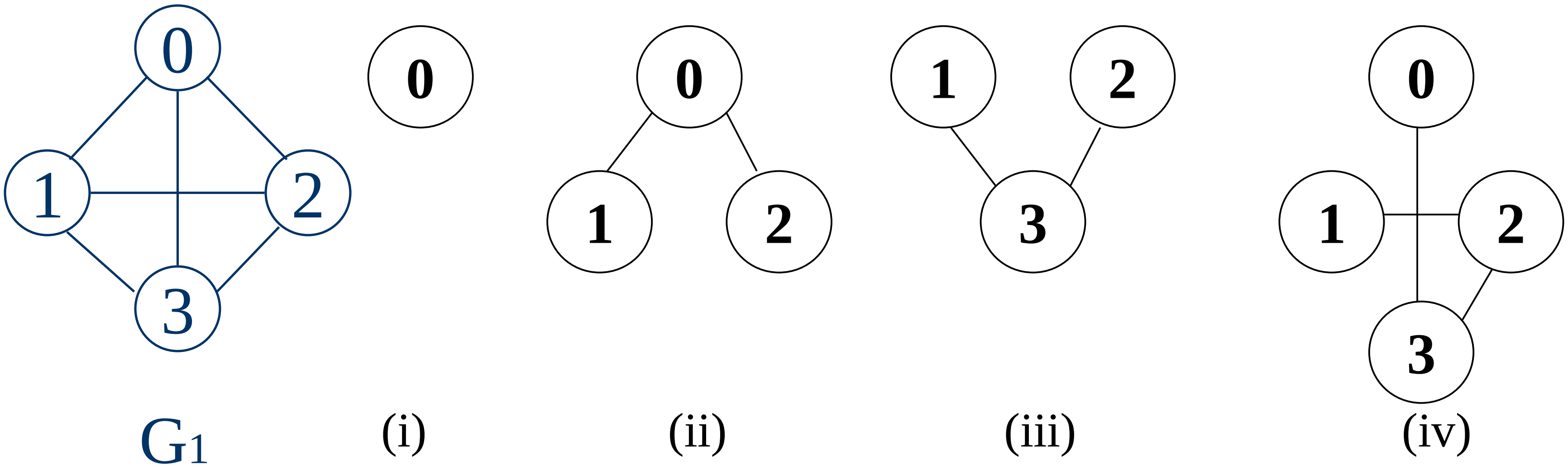
(b) multigraph

multiple occurrences of the same edge

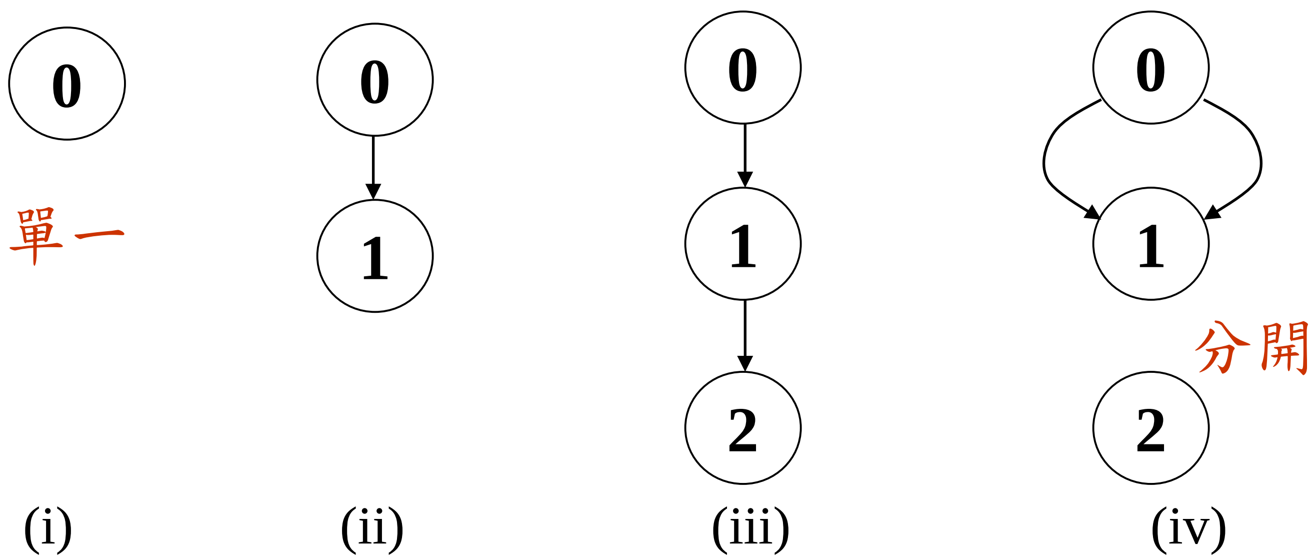
Subgraph and Path

- A subgraph of G is a graph G' such that V(G') is a subset of V(G) and E(G') is a subset of E(G)
- A path from vertex v_p to vertex v_q in a graph G, is a sequence of vertices, $v_p, v_{i1}, v_{i2}, \ldots, v_{in}, v_q$, such that $(v_p, v_{i1}), (v_{i1}, v_{i2}), \ldots, (v_{in}, v_q)$ are edges in an undirected graph
- The length of a path is the number of edges on it

Figure 6.4: subgraphs of G_1 and G_3



(a) Some of the subgraph of G_1

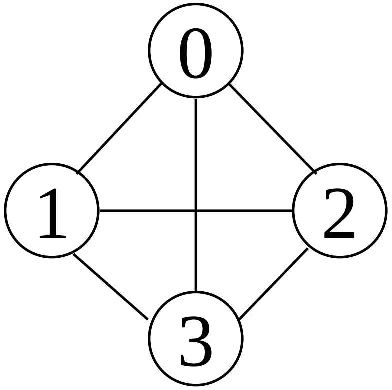


(b) Some of the subgraph of G_3

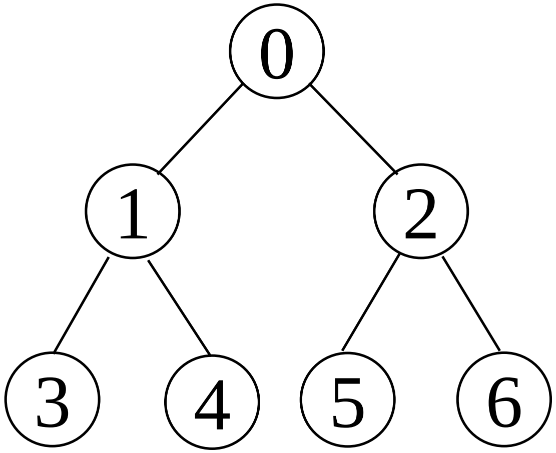
Simple Path and Style

- A simple path is a path in which all vertices, except possibly the first and the last, are distinct
- A cycle is a simple path in which the first and the last vertices are the same
- In an undirected graph G, two vertices, v_0 and v_1, are connected iff there is a path in G from v_0 to v_1
- An undirected graph is connected iff for every pair of distinct vertices v_i, v_j, there is a path from v_i to v_j

Connected



G_1



G_2

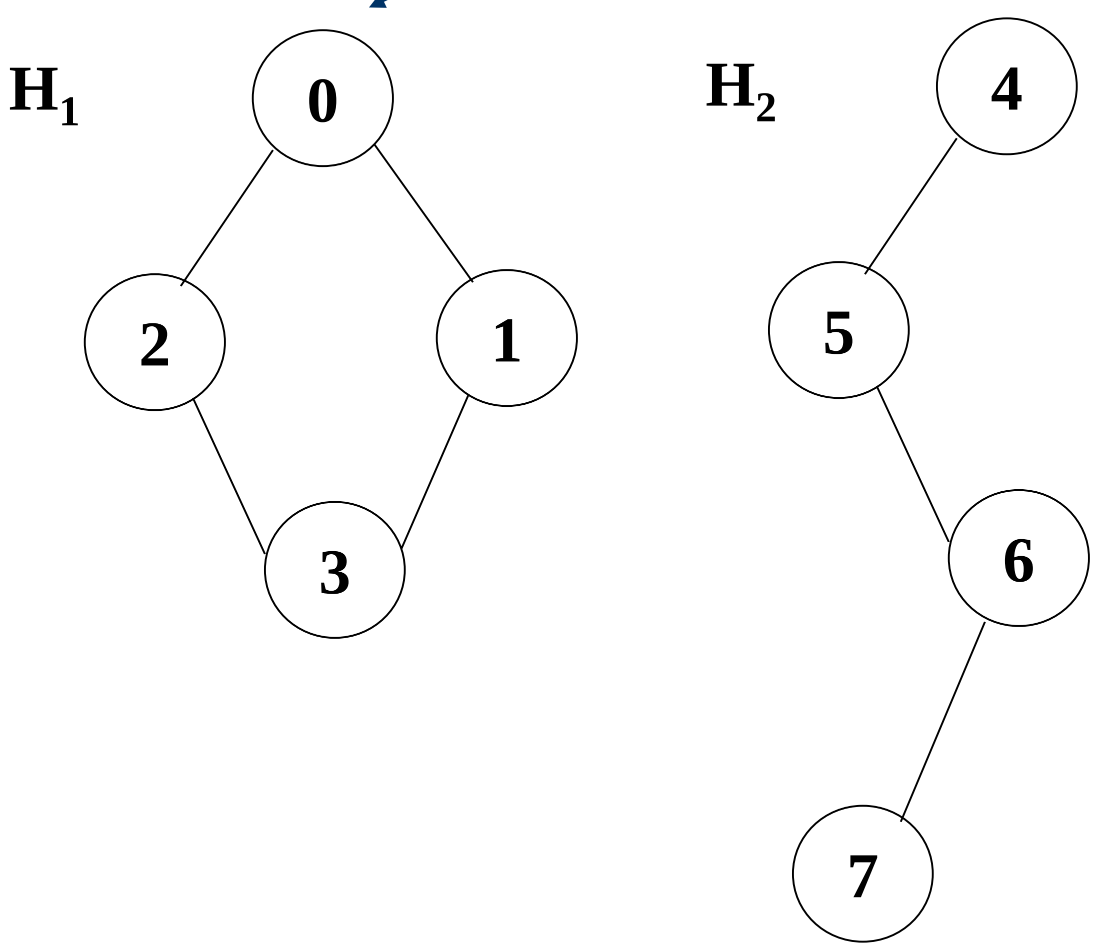
tree (acyclic graph)

Connected Component

- A connected component of an undirected graph is a maximal connected subgraph.
- A tree is a graph that is connected and acyclic (i.e., has no cycles).
- A directed graph is strongly connected if there is a directed path from v_i to v_j and also from v_j to v_i.
- A strongly connected component is a maximal subgraph that is strongly connected.

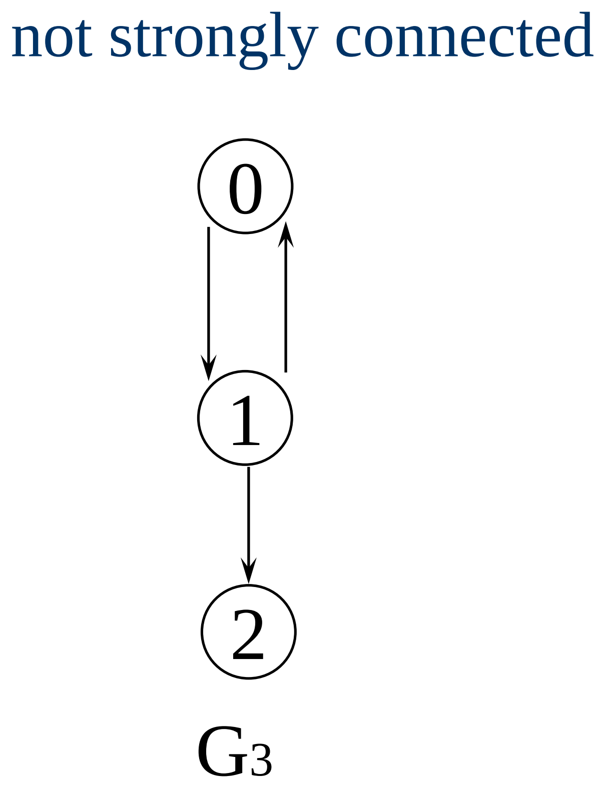
***Figure 6.5:** A graph with two connected components (p.262)

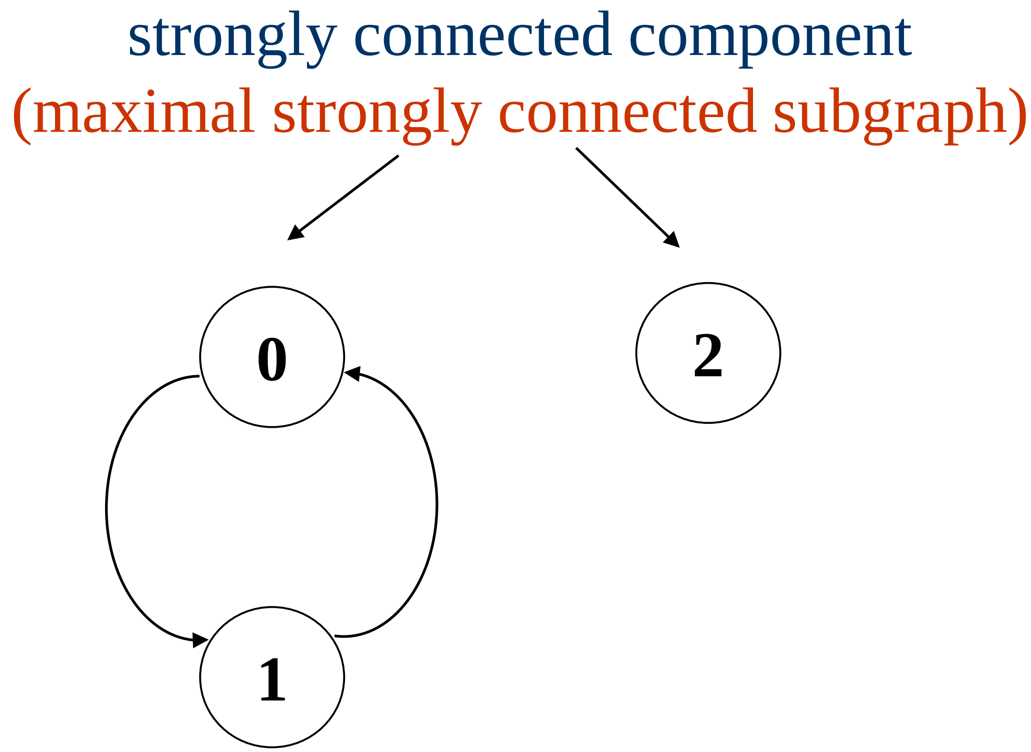
connected component (maximal connected subgraph)



G_4 (not connected)

***Figure 6.6:** Strongly connected components of G_3





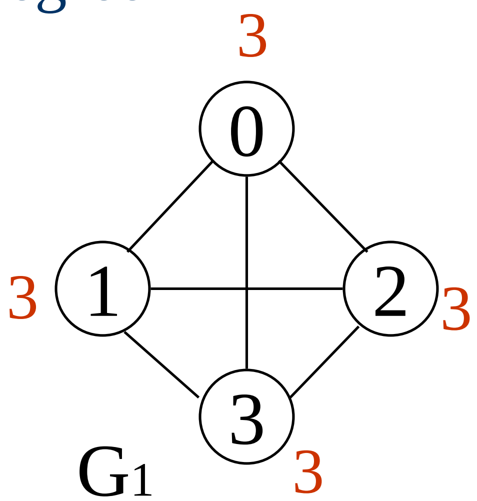
Degree

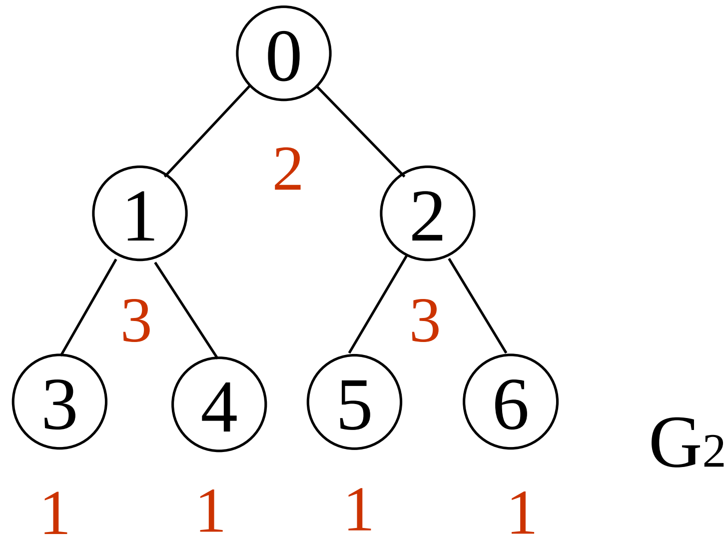
- The degree of a vertex is the number of edges incident to that vertex
- For directed graph,
  - the in-degree of a vertex v is the number of edges that have v as the head
  - the out-degree of a vertex v is the number of edges that have v as the tail
  - if d_i is the degree of a vertex i in a graph G with n vertices and e edges, the number of edges is

$$e = \left(\sum_{0}^{n-1} d_i\right) / 2$$

undirected graph

degree

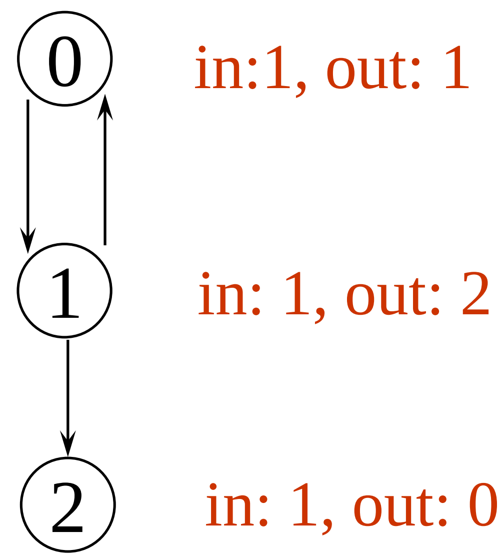




directed graph

in-degree

out-degree



G_3

ADT for Graph

structure Graph is

objects: a nonempty set of vertices and a set of undirected edges, where each edge is a pair of vertices

functions: for all *graph* $\in$ *Graph*, *v*, v_1 and $v_2 \in$ *Vertices*

Graph Create()::=return an empty graph

Graph InsertVertex(*graph*, *v*)::= return a graph with *v* inserted. *v* has no incident edge.

Graph InsertEdge(*graph*, v_1,v_2)::= return a graph with new edge between v_1 and v_2

Graph DeleteVertex(*graph*, *v*)::= return a graph in which *v* and all edges incident to it are removed

Graph DeleteEdge(*graph*, v_1, v_2)::=return a graph in which the edge (v_1, v_2) is removed

Boolean IsEmpty(*graph*)::= if (*graph*==*empty graph*) return TRUE else return FALSE

List Adjacent(*graph*,*v*)::= return a list of all vertices that are adjacent to *v*

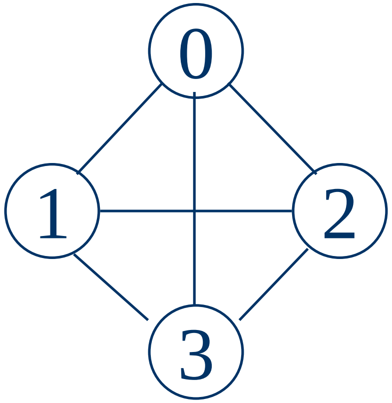
Graph Representations

- Adjacency Matrix
- Adjacency Lists
- Adjacency Multilists

Adjacency Matrix

- Let G=(V,E) be a graph with *n* vertices.
- The adjacency matrix of G is a two-dimensional *n* n* array, say adj_mat
- If the edge (v_i, v_j) is in E(G), adj_mat[i][j]=1
- If there is no such edge in E(G), adj_mat[i][j]=0
- The adjacency matrix for an undirected graph is symmetric; the adjacency matrix for a digraph need not be symmetric

Examples for Adjacency Matrix

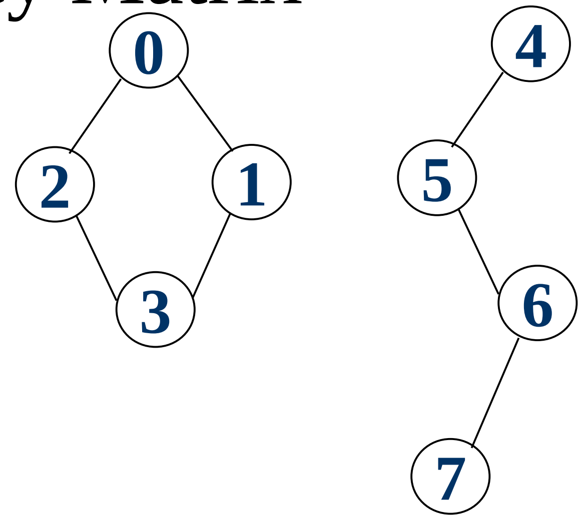


$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

G_1

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

G_2



$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

G_4

symmetric

undirected: $n^2/2$
directed: n^2

Merits of Adjacency Matrix

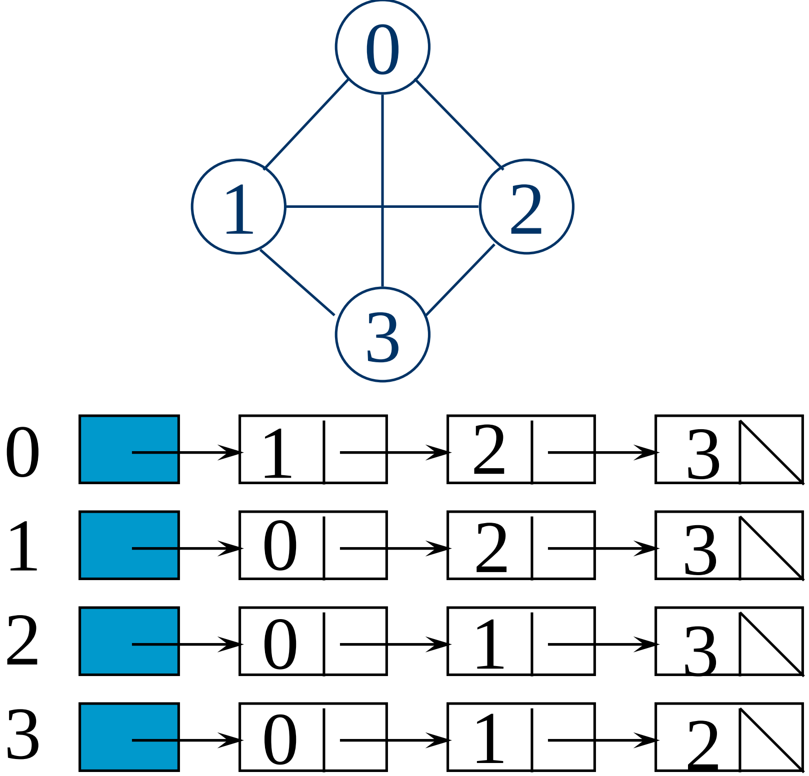
- From the adjacency matrix, to determine the connection of vertices is easy
- The degree of a vertex is $\sum_{j=0}^{n-1} adj\_mat[i][j]$
- For a directed graph, the row sum is the out_degree, while the column sum is the in_degree

$$ind(vi) = \sum_{j=0}^{n-1} A[j,i] \qquad outd(vi) = \sum_{j=0}^{n-1} A[i,j]$$

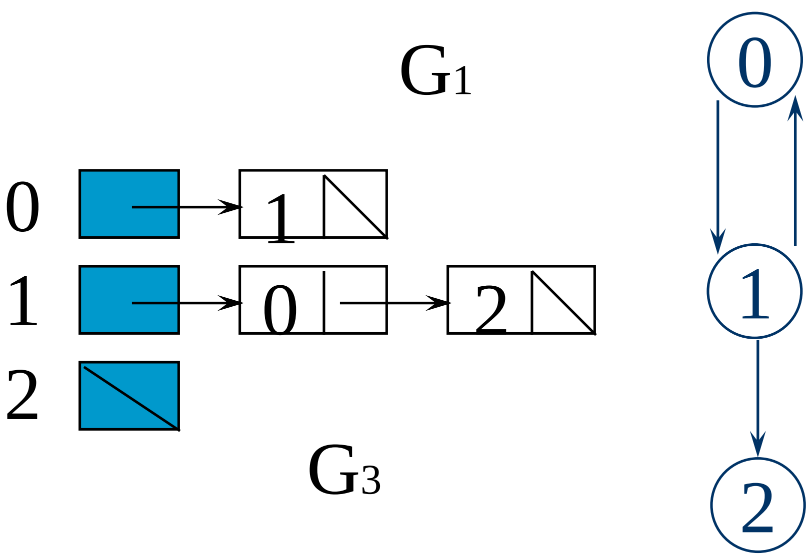
Data Structures for Adjacency Lists

Each row in adjacency matrix is represented as an adjacency list.

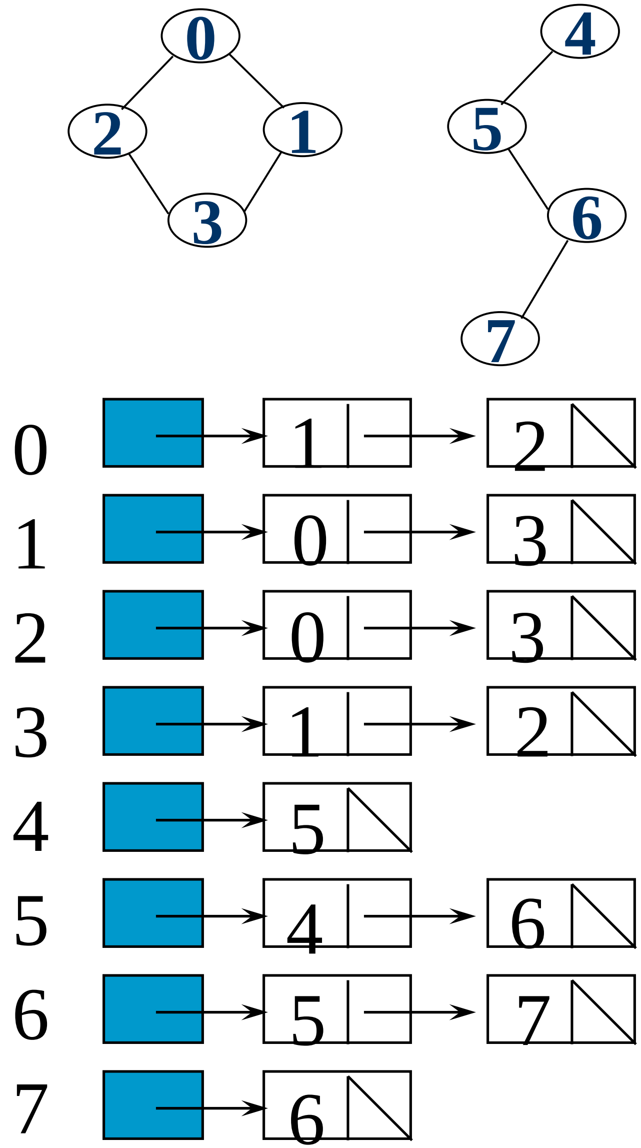
```
#define MAX_VERTICES 50
typedef struct node *node_pointer;
typedef struct node {
    int vertex;
    struct node *link;
};
node_pointer graph[MAX_VERTICES];
int n=0; /* vertices currently in use */
```



G_1



G_3

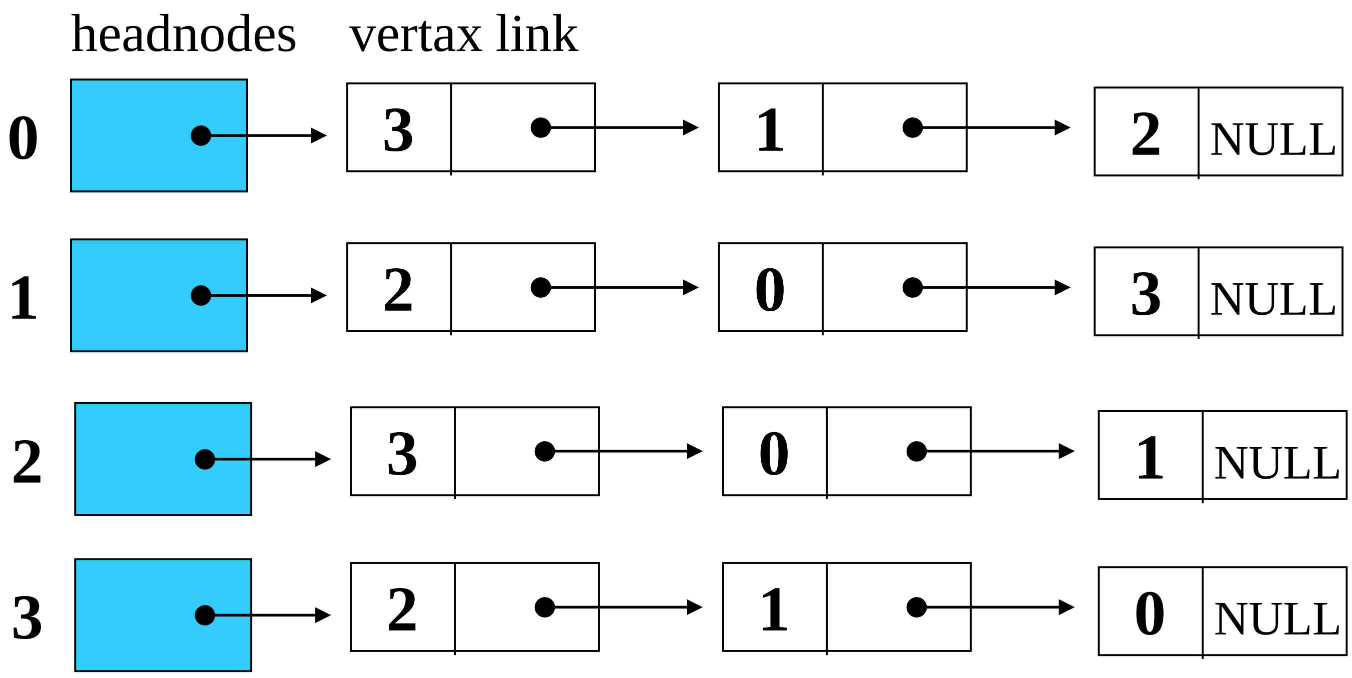


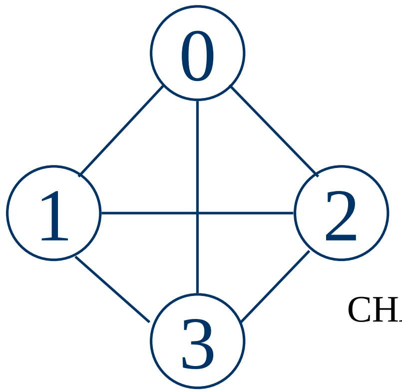
G_4

An undirected graph with *n* vertices and *e* edges ==> n head nodes and 2e list nodes

Alternate order adjacency list for G_1

Order is of no significance.



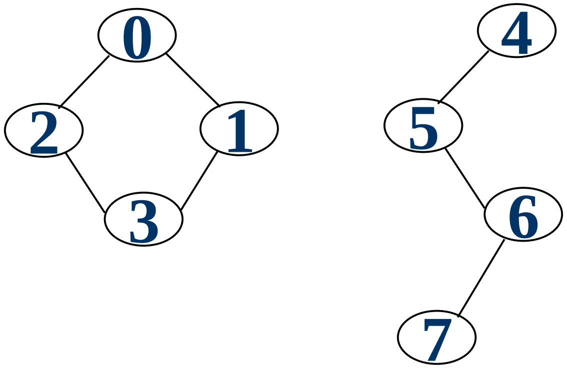


Interesting Operations

- degree of a vertex in an undirected graph
  – # of nodes in adjacency list
- # of edges in a graph
  – determined in O(n+e)
- out-degree of a vertex in a directed graph
  – # of nodes in its adjacency list
- in-degree of a vertex in a directed graph
  – traverse the whole data structure

An undirected graph with *n* vertices and *e* edges

Compact Representation



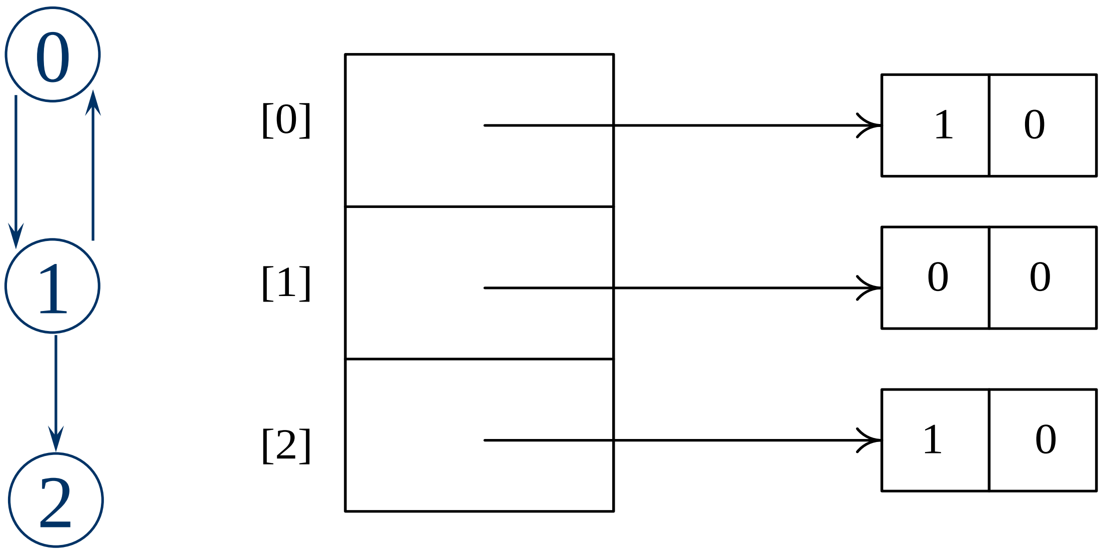
node[0] … node[n-1]: starting point for vertices

node[n]: n+2e+1

node[n+1] … node[n+2e]: head node of edge

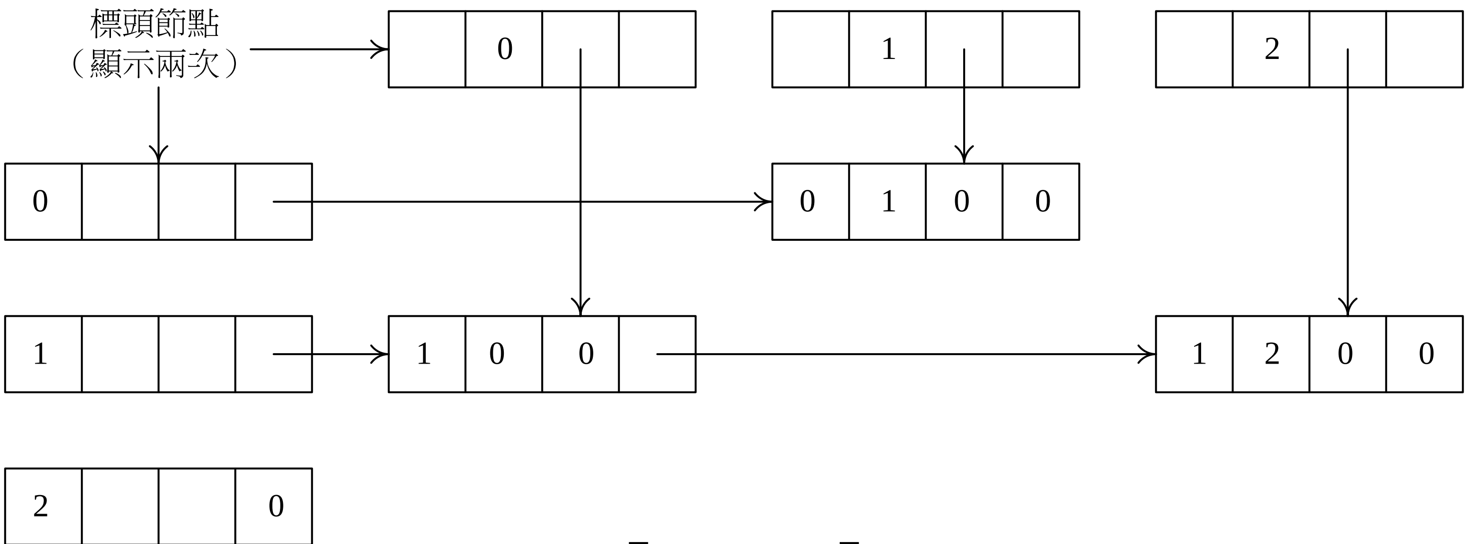
[0]	9		[8]	23		[16]	2
[1]	11	0	[9]	1	4	[17]	5
[2]	13		[10]	2	5	[18]	4
[3]	15	1	[11]	0		[19]	6
[4]	17		[12]	3	6	[20]	5
[5]	18	2	[13]	0		[21]	7
[6]	20		[14]	3	7	[22]	6
[7]	22	3	[15]	1			

Figure 6.10: Inverse adjacency list for G_3

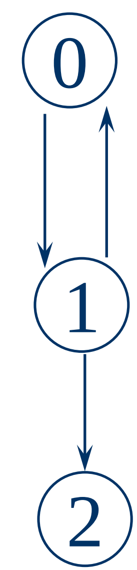


Determine in-degree of a vertex in a fast way.

Figure 6.11: Orthogonal representation for graph



$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$



row	col	column link for head	row link for tail

Adjacency Multilists

- An edge in an undirected graph is represented by two nodes in adjacency list representation.
- Adjacency Multilists

–lists in which nodes may be shared among several lists.

(**an edge is shared by two different paths**)

marked	vertex1	vertex2	path1	path2

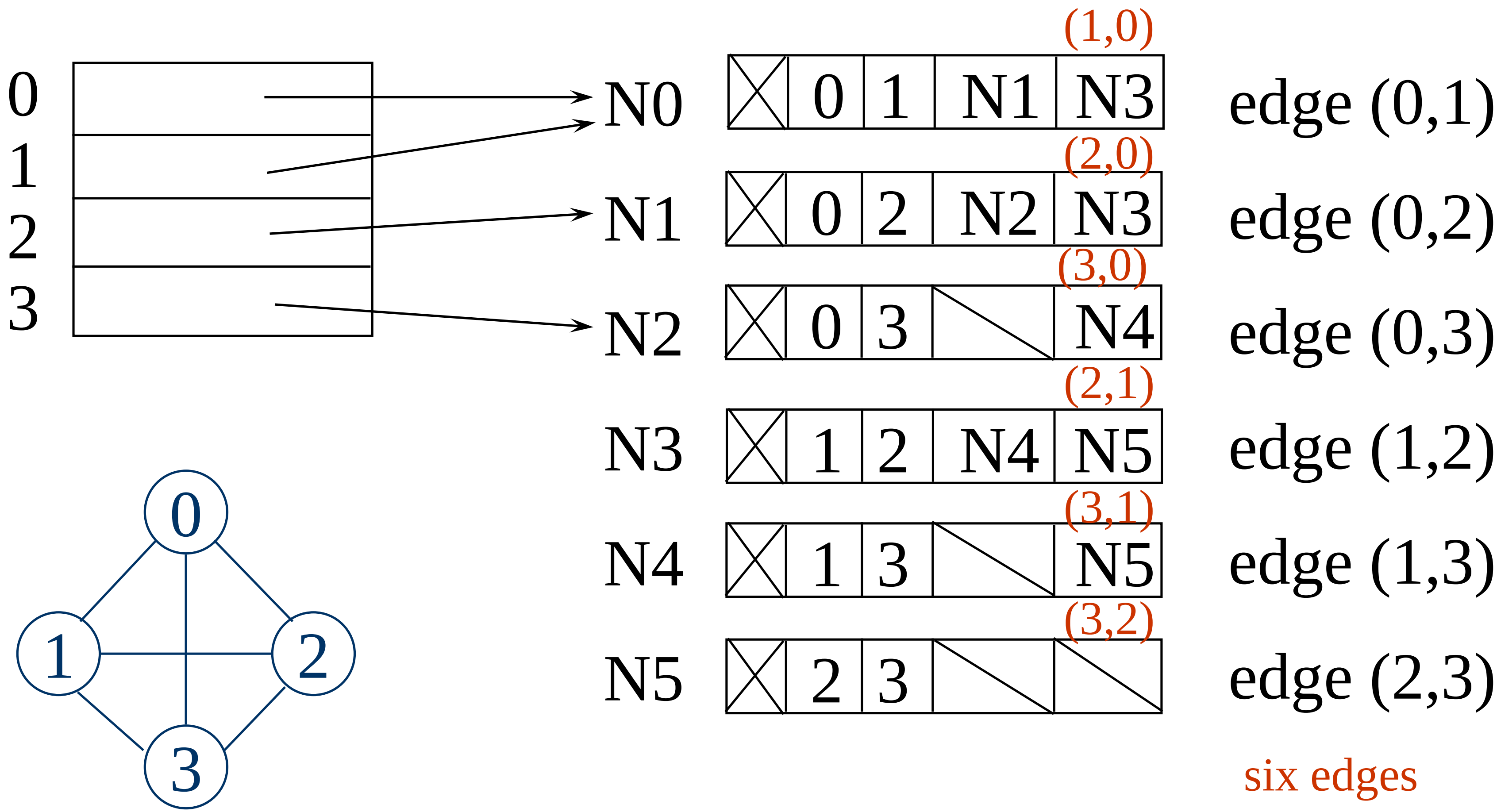
Adjacency Multilists

```
typedef struct edge *edge_pointer;
typedef struct edge {
    short int marked;
    int vertex1, vertex2;
    edge_pointer path1, path2;
};
edge_pointer graph[MAX_VERTICES];
```

marked	vertex1	vertex2	path1	path2

Example for Adjacency Multlists

Lists: vertex 0: N0->N1->N2, vertex 1: N0->N3->N4
vertex 2: N1->N3->N5, vertex 3: N2->N4->N5



six edges

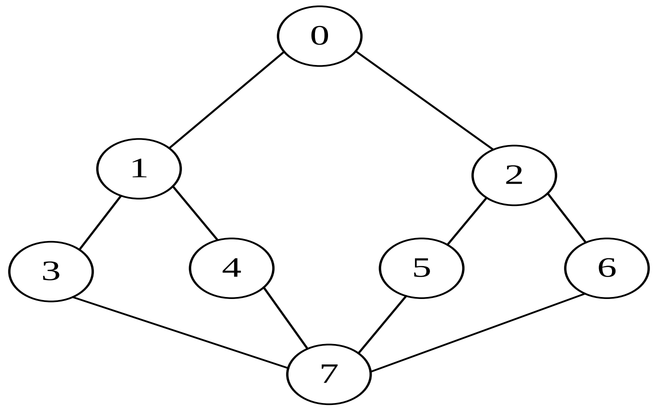
Some Graph Operations

- Traversal
  Given G=(V,E) and vertex v, find all $w \in V$, such that w connects v.
  - Depth First Search (DFS)
    preorder tree traversal
  - Breadth First Search (BFS)
    level order tree traversal
- Connected Components
- Spanning Trees

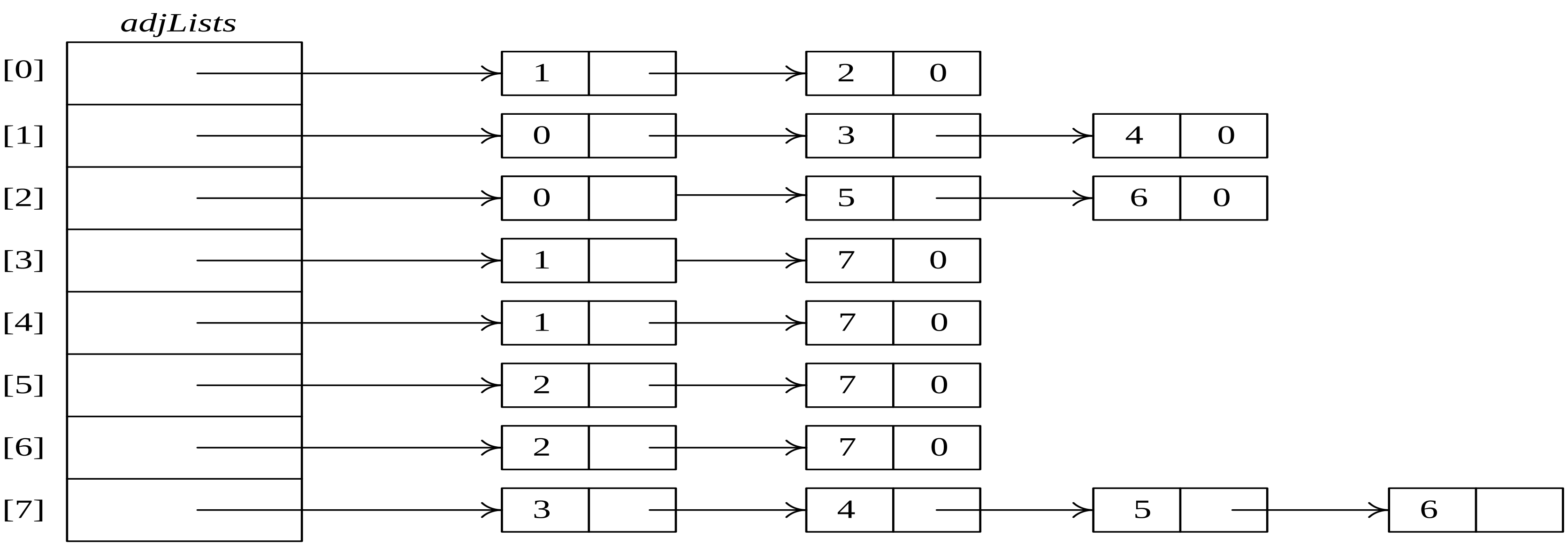
***Figure 6.16:** Graph *G* and its adjacency lists

depth first search: v0, v1, v3, v7, v4, v5, v2, v6

breadth first search: v0, v1, v2, v3, v4, v5, v6, v7



(a)



(b)

Depth First Search

```
#define FALSE 0
#define TRUE 1
short int visited[MAX_VERTICES];
```

```
void dfs(int v)
{
  node_pointer w;
  visited[v]= TRUE;
  printf("%5d", v);
  for (w=graph[v]; w; w=w->link)
    if (!visited[w->vertex])
      dfs(w->vertex);
}
```

Data structure

adjacency list: O(e)

adjacency matrix: $O(n^2)$

Breadth First Search

```
typedef struct queue *queue_pointer;
typedef struct queue {
    int vertex;
    queue_pointer link;
};
void addq(int);
int deleteq();
```

Breadth First Search *(Continued)*

```
void bfs(int v)
{
  node_pointer w;
  queue_pointer front, rear;
  front = rear = NULL;
  printf("%5d", v);
  visited[v] = TRUE;
  addq(v);
```

adjacency list: O(e)
adjacency matrix: O(n^2)

```
  while (front) {
    v= deleteq();
    for (w=graph[v]; w; w=w->link)
      if (!visited[w->vertex]) {
        printf(“%5d”, w->vertex);
        addq(w->vertex);
        visited[w->vertex] = TRUE;
      }/* unvisited vertices*/
  }
}
```

Connected Components

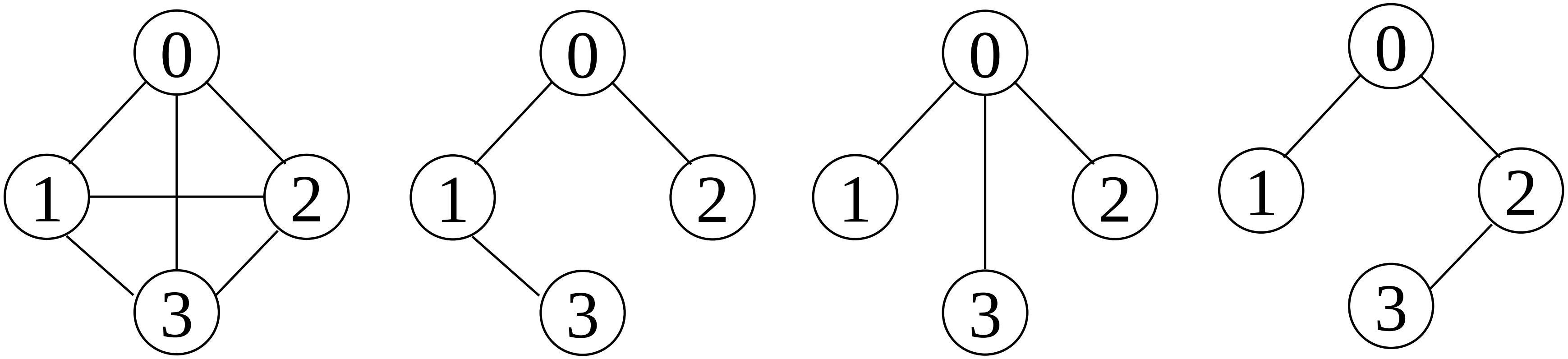
```
void connected(void)
{ /*determine the connected components of
  a graph */
    for (i=0; i<n; i++) {
        if (!visited[i]) {
            dfs(i); // dfs→O(n)
            printf(“\n”);
        }
    }
}
```

adjacency list: $O(n+e)$
adjacency matrix: $O(n^2)$

Spanning Trees

- When graph G is connected, a depth first or breadth first search starting at any vertex will visit all vertices in G
- A spanning tree is any tree that consists solely of edges in G and that includes all the vertices
- E(G): T (tree edges) + N (nontree edges)
  where T: set of edges used during search
  N: set of remaining edges

Examples of Spanning Tree



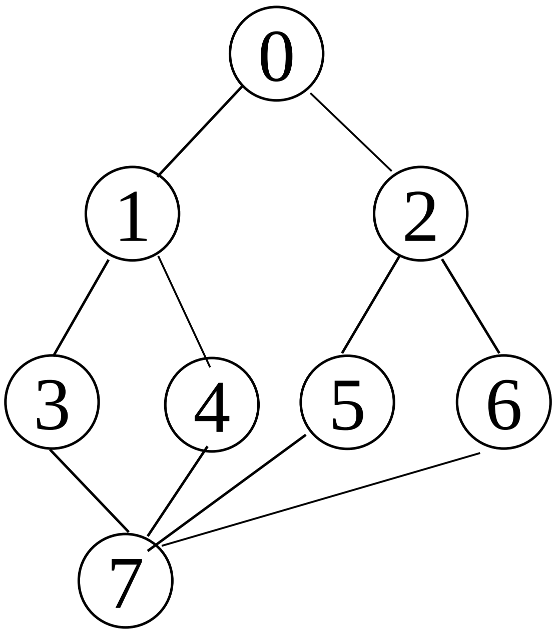
G_1

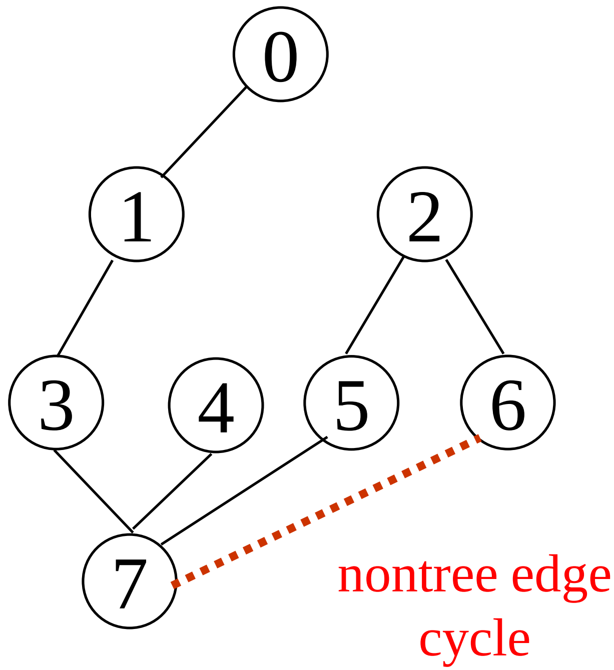
Possible spanning trees

Spanning Trees

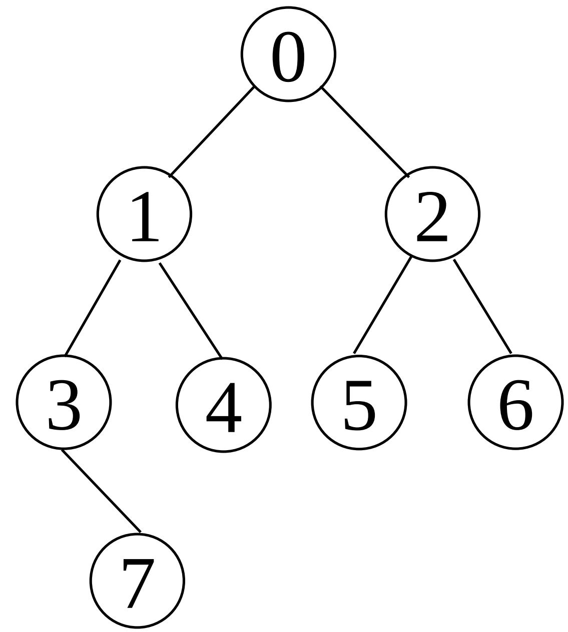
- Either dfs or bfs can be used to create a spanning tree
  - When *dfs* is used, the resulting spanning tree is known as a **depth first spanning tree**
  - When *bfs* is used, the resulting spanning tree is known as a **breadth first spanning tree**
- While adding a nontree edge into any spanning tree, this will create a cycle

DFS vs BFS Spanning Tree





DFS Spanning

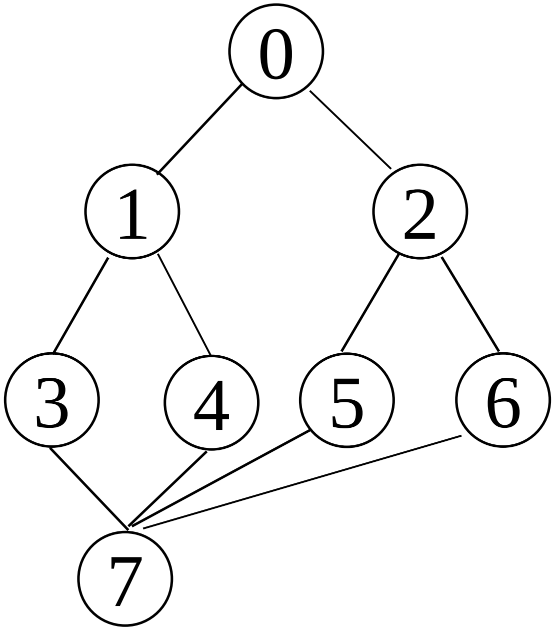


BFS Spanning

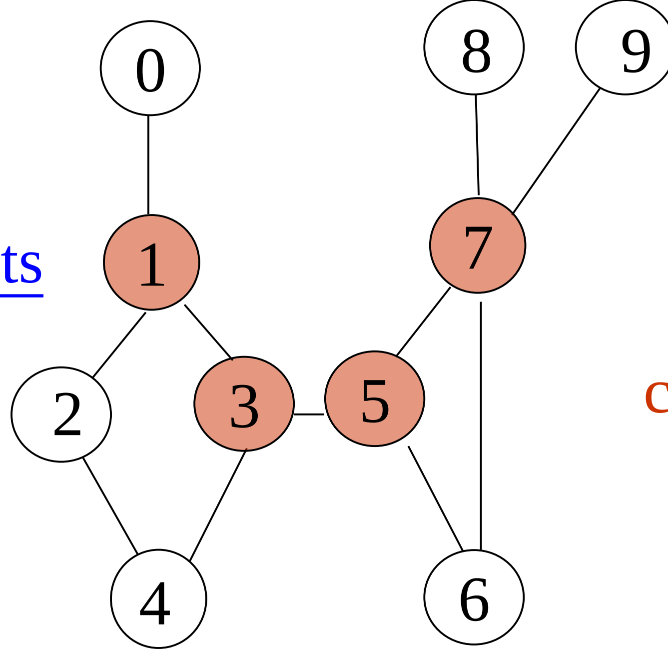
A spanning tree is a minimal subgraph, G', of G such that V(G')=V(G) and G' is connected.

Any connected graph with *n* vertices must have at least *n-1* edges.

A biconnected graph is a connected graph that has no articulation points.

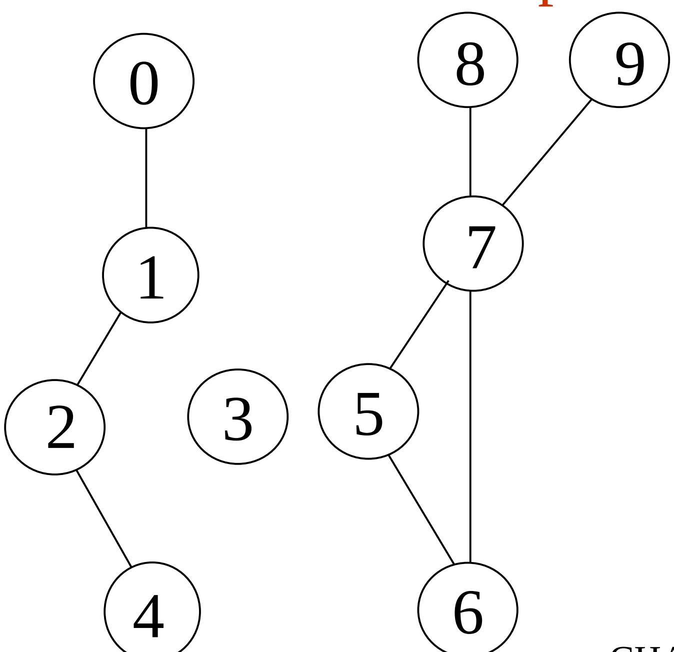


Articulation points

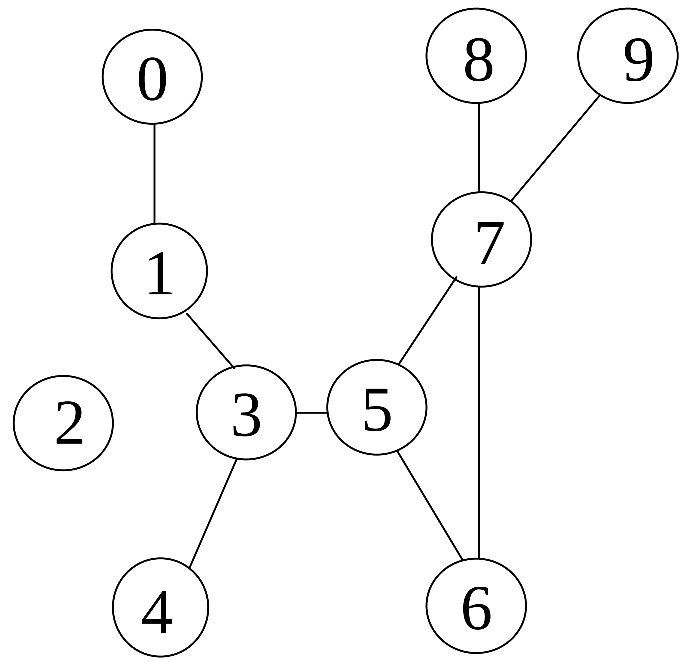


connected graph

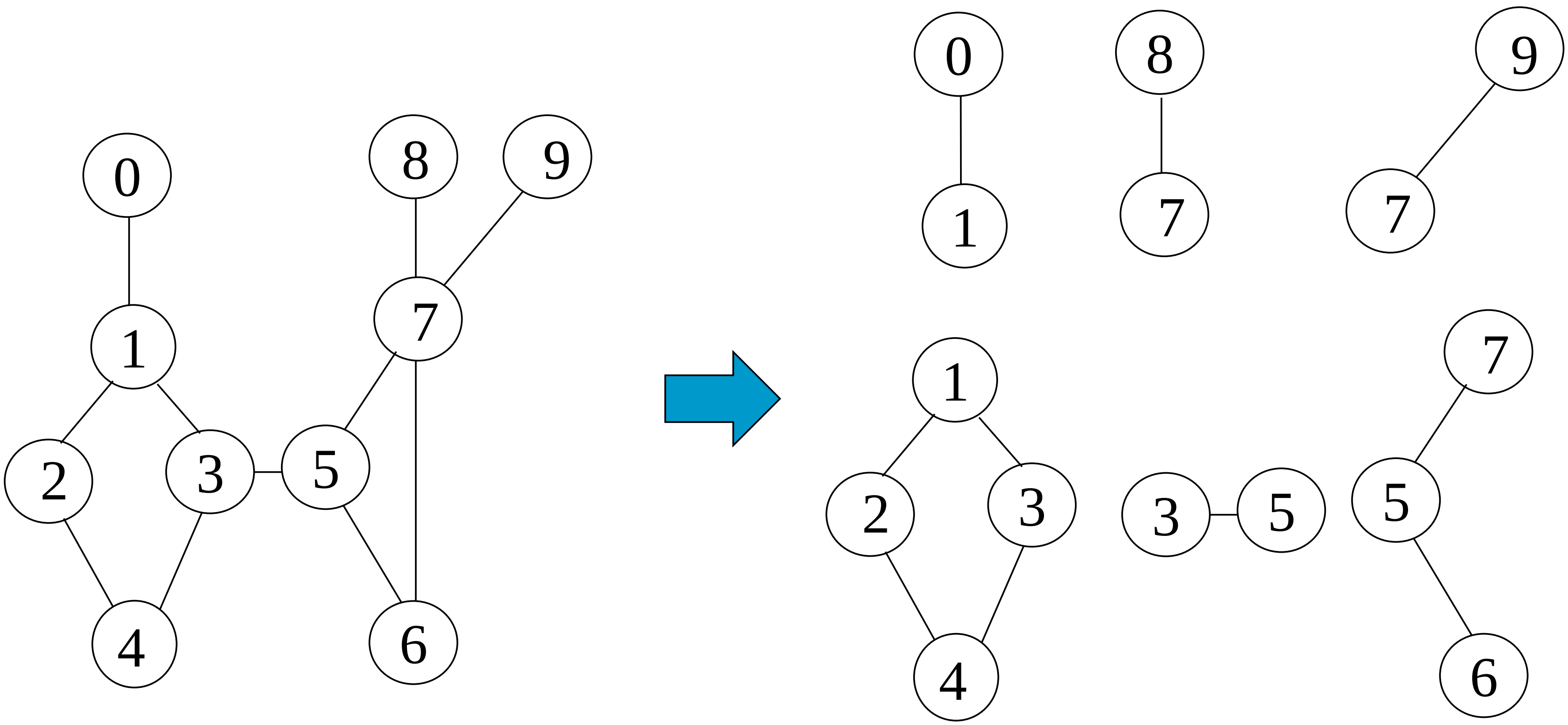
two connected components



one connected graph



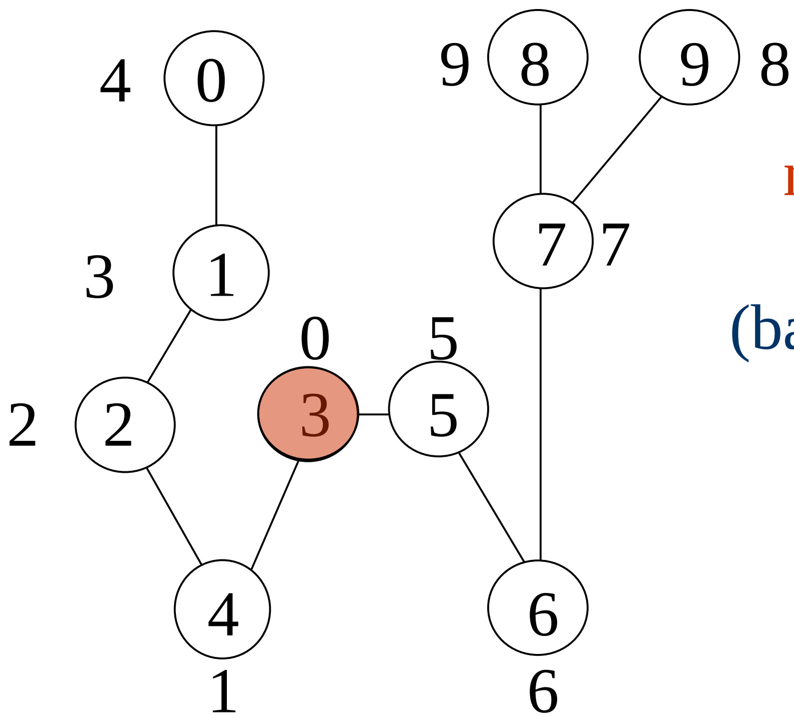
biconnected component: a maximal connected subgraph H
(no subgraph that is both biconnected and properly contains H)



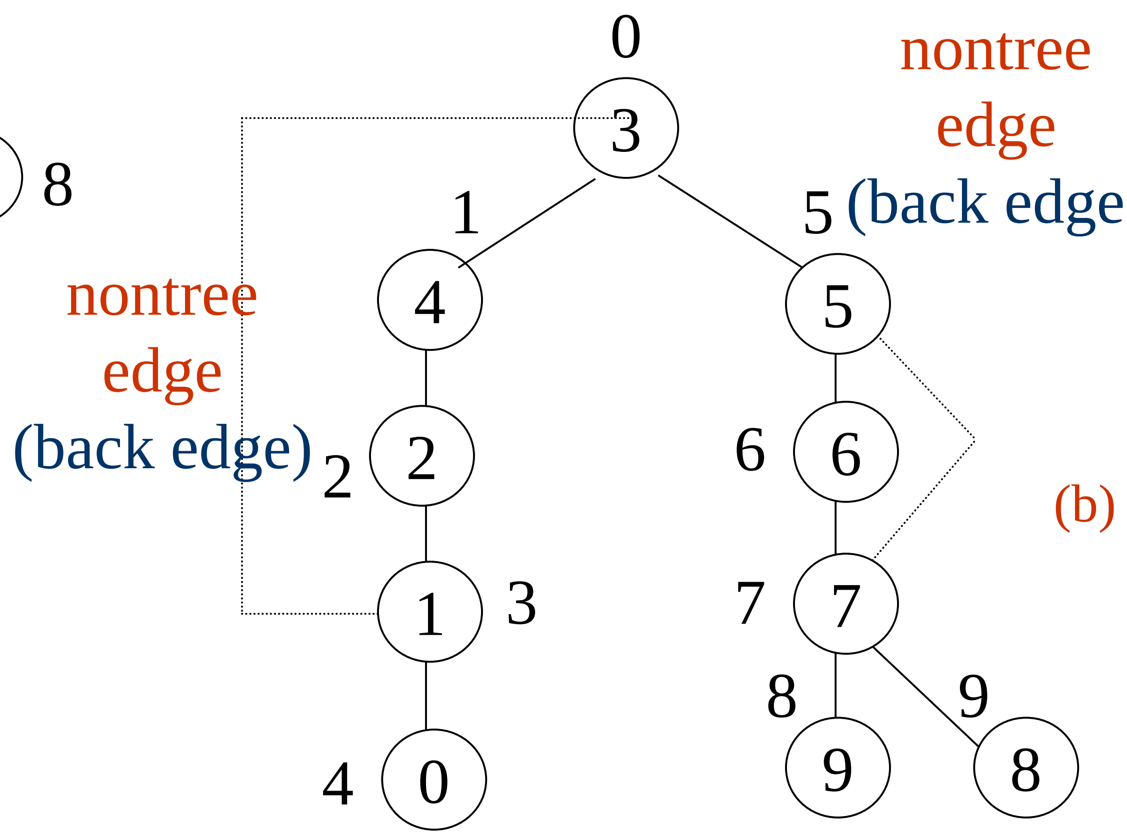
biconnected components

Find biconnected component of a connected undirected graph by depth first spanning tree

depth first number (dfn)



(a) depth first spanning tree



Any other vertex u is an articulation point iff it has at least one child w such that we cannot reach an ancestor of u using a path

If u is an ancestor of v then dfn(u) < dfn(v).

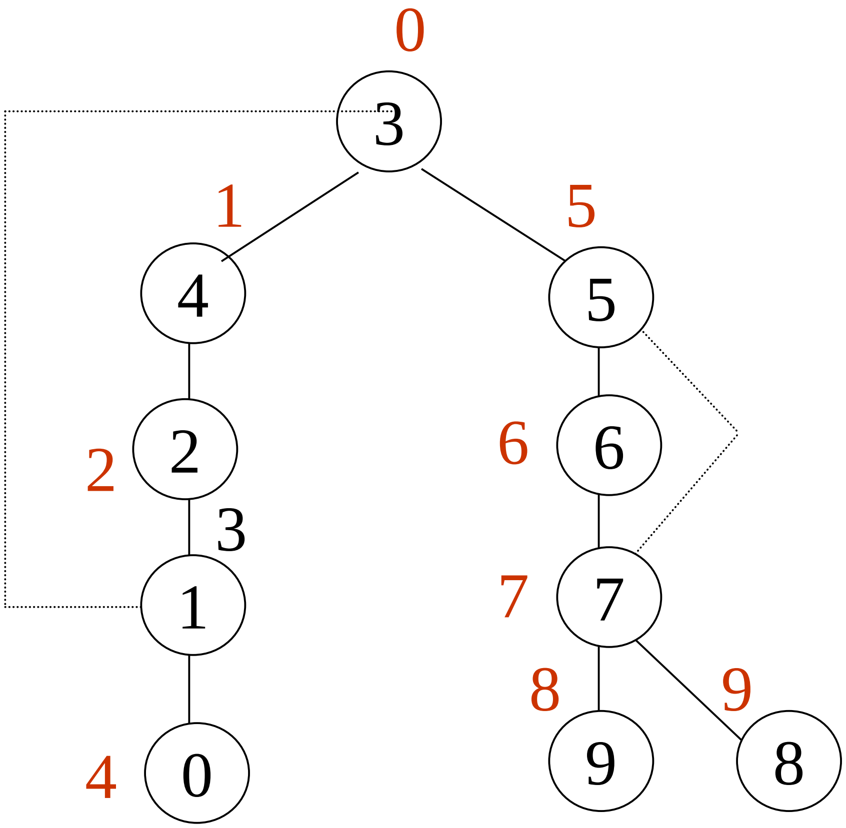
***Figure 6.21**: *dfn* and *low* values for *dfs* spanning tree with *root* =3

Vertax	0	1	2	3	4	5	6	7	8	9
dfn	4	3	2	0	1	5	6	7	9	8
low	4	0	0	0	0	5	5	5	9	8

low(u)=min{dfn(u), min{low(w)|w is a child of u},
min{dfn(w)|(u,w) is a back edge}

u: articulation point

low(child) ≥ dfn(u)



*The root of a depth first spanning tree is an articulation point iff it has at least two children.

*Any other vertex u is an articulation point iff it has at least one child w such that we cannot reach an ancestor of u using a path that consists of

(1) only w;

(2) descendants of w;

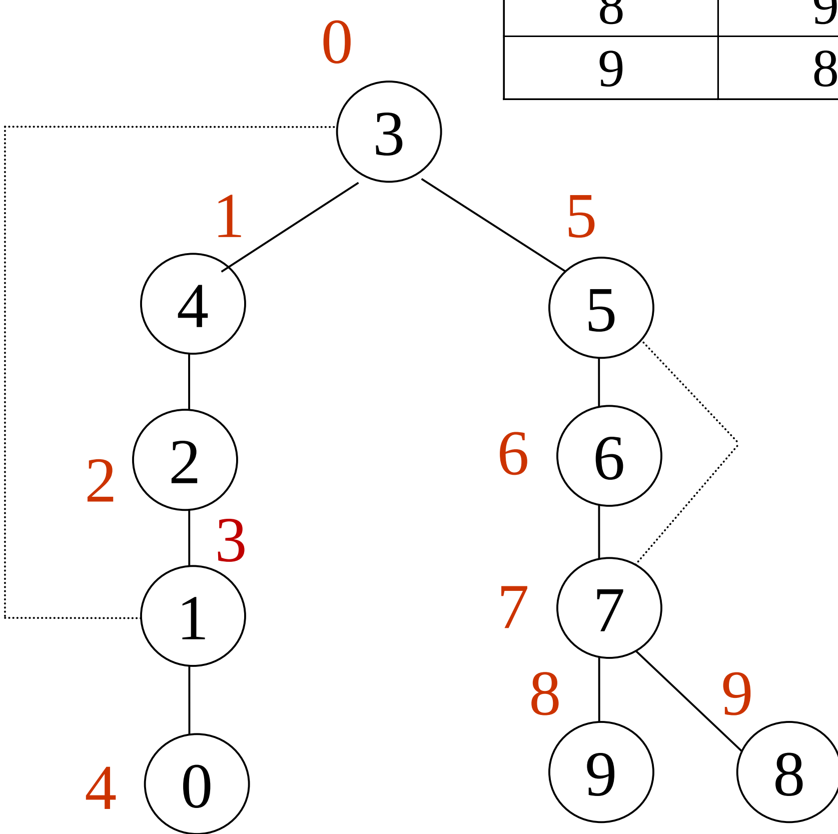
(3) single back edge.

low(u)=min{dfn(u), min{low(w)|w is a child of u},
min{dfn(w)|(u,w) is a back edge}

u: articulation point

low(child) ≥ dfn(u)

vertex	dfn	low	child	low_child	low:dfn
0	4	4 (4,n,n)	null	null	null:4
1	3	0 (3,4,0)	0	4	4 ≥ 3 ●
2	2	0 (2,0,n)	1	0	0 < 2
3	0	0 (0,0,n)	4,5	0,5	0,5 ≥ 0 ●
4	1	0 (1,0,n)	2	0	0 < 1
5	5	5 (5,5,n)	6	5	5 ≥ 5 ●
6	6	5 (6,5,n)	7	5	5 < 6
7	7	5 (7,8,5)	8,9	9,8	9,8 ≥ 7 ●
8	9	9 (9,n,n)	null	null	null, 9
9	8	8 (8,n,n)	null	null	null, 8



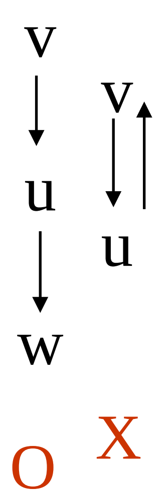
low(u)=min{dfn(u),
min{low(w)|w is a child of u},
min{dfn(w)|(u,w) is a back edge}

```
void init(void)
 {
   int i;
   for (i = 0; i < n; i++) {
        visited[i] = FALSE;
        dfn[i] = low[i] = -1;
        }
        num = 0;
  }
```

***Program 6.5:** Initializaiton of *dfn* and *low*

*Program 6.4: Determining *dfn* and *low*

```
void dfnlow(int u, int v)                    Initial call: dfn(x,-1)
{
/* compute dfn and low while performing a dfs search
   beginning at vertex u, v is the parent of u (if any) */
      node_pointer ptr;
      int w;
      dfn[u] = low[u] = num++;              low[u]=min{dfn(u), …}
      for (ptr = graph[u]; ptr; ptr = ptr ->link) {
         w = ptr ->vertex;
         if (dfn[w] < 0) { /*w is an unvisited vertex */
            dfnlow(w, u);
            low[u] = MIN2(low[u], low[w]);
         }    low[u]=min{…, min{low(w)|w is a child of u}, …}
         else if (w != v)  dfn[w]≠0 非第一次，表示藉back edge
           low[u] =MIN2(low[u], dfn[w] );
   }
}        low[u]=min{…,…,min{dfn(w)|(u,w) is a back edge}
```



***Program 6.6:** Biconnected components of a graph

```
void bicon(int u, int v)
{
/* compute dfn and low, and output the edges of G by their
   biconnected  components , v is the parent ( if any) of the u
  (if any)  in the resulting spanning tree. It is assumed that all
  entries  of dfn[ ] have been initialized to -1, num has been
  initialized to 0,  and the stack has been set to empty */
    node_pointer ptr;
    int w, x, y;
    dfn[u] = low[u] = num ++;
    for (ptr = graph[u]; ptr; ptr = ptr->link) {
       w = ptr ->vertex;
       if ( v != w && dfn[w] < dfn[u] )
          push(u, w);        /* add edge to stack */
```

low[u]=min{dfn(u), …}

(1) dfn[w]=-1 第一次

(2) dfn[w]!=-1非第一次，藉back edge

```
if(dfn[w] < 0) {/* w has not been visited */
    bicon(w, u); low[u]=min{…, min{low(w)|w is a child of u}, .
    low[u] = MIN2(low[u], low[w]);
    if (low[w] >= dfn[u] ){     articulation point
        printf("New biconnected component: ");
        do { /* delete edge from stack */
            pop(&x, &y);
            printf(" <%d, %d>" , x, y);
        } while (!(( x = = u) && (y = = w)));
        printf("\n");
    }
  }
  else if (w != v)  low[u] = MIN2(low[u], dfn[w]);
 }  low[u]=min{…, …, min{dfn(w)|(u,w) is a back edge}}
}
```

Minimum Cost Spanning Tree

- The cost of a spanning tree of a weighted undirected graph is the sum of the costs of the edges in the spanning tree
- A minimum cost spanning tree is a spanning tree of least cost
- Three different algorithms can be used
  – Kruskal
  – Prim
  – Sollin

Select *n-1* edges from a weighted graph of *n* vertices with minimum cost.

Greedy Strategy

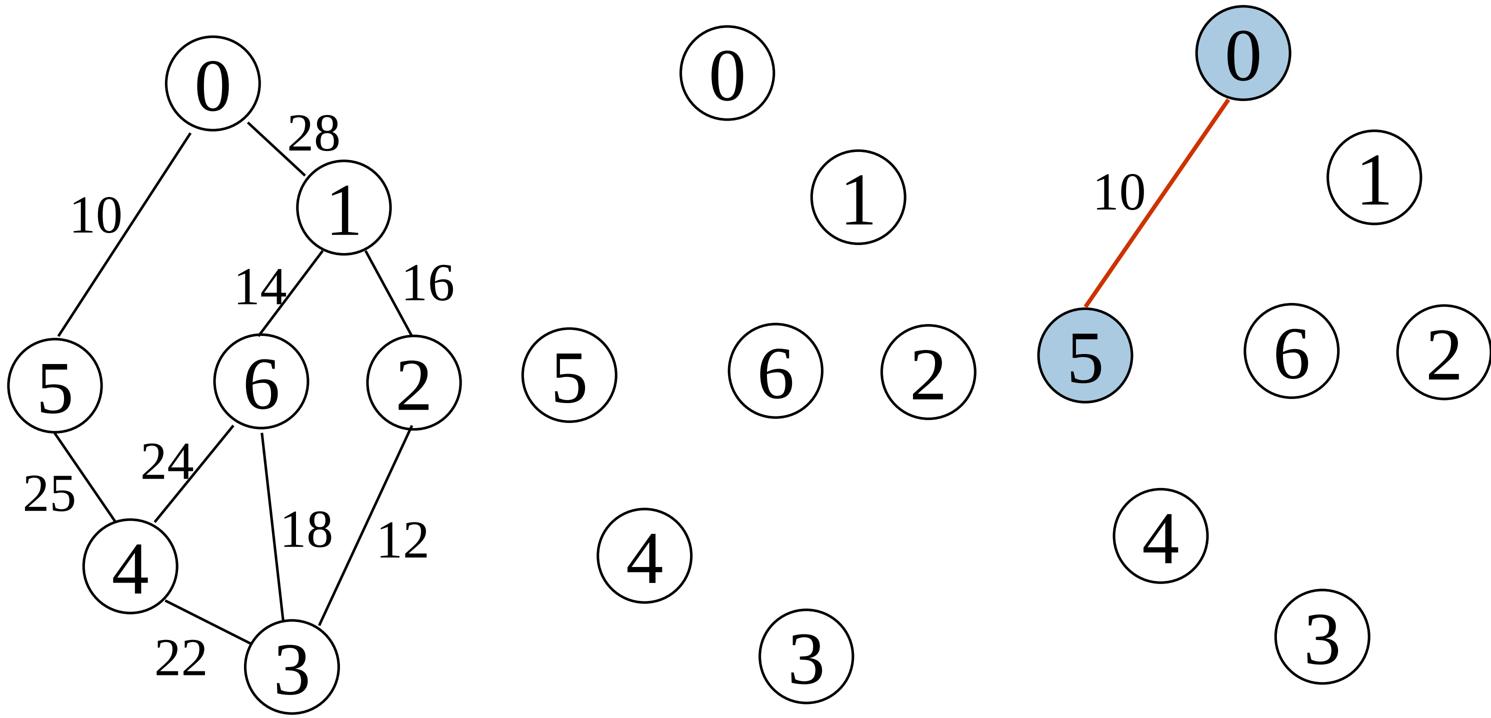
- An optimal solution is constructed in stages
- At each stage, the best decision is made at this time
- Since this decision cannot be changed later, we make sure that the decision will result in a feasible solution
- Typically, the selection of an item at each stage is based on a least cost or a highest profit criterion

Kruskal's Idea

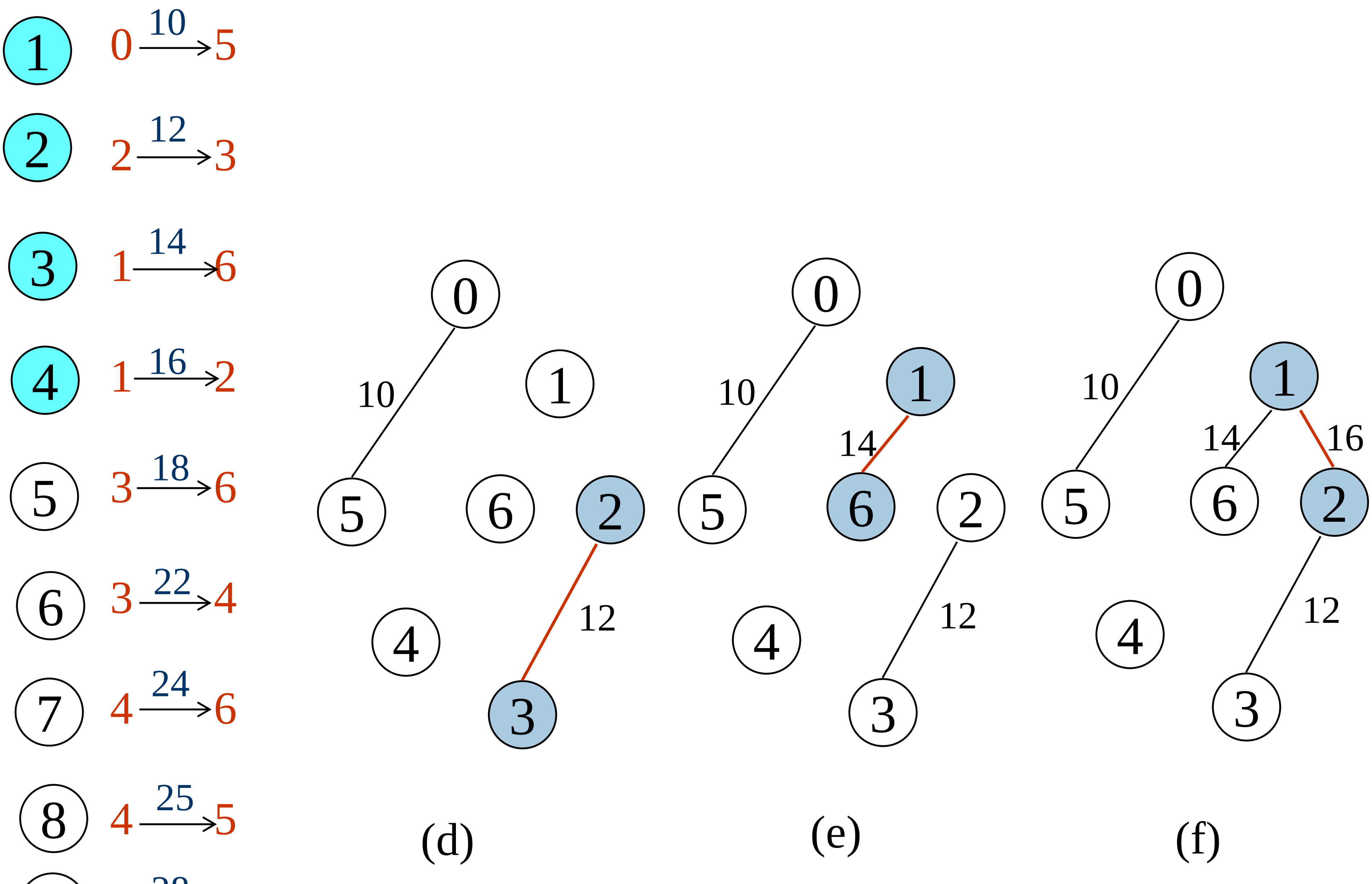
- Build a minimum cost spanning tree T by adding edges to T one at a time
- Select the edges for inclusion in T in nondecreasing order of the cost
- An edge is added to T if it does not form a cycle
- Since G is connected and has n > 0 vertices, exactly n-1 edges will be selected

Examples for Kruskal's Algorithm

1. $0 \xrightarrow{10} 5$
2. $2 \xrightarrow{12} 3$
3. $1 \xrightarrow{14} 6$
4. $1 \xrightarrow{16} 2$
5. $3 \xrightarrow{18} 6$
6. $3 \xrightarrow{22} 4$
7. $4 \xrightarrow{24} 6$
8. $4 \xrightarrow{25} 5$
9. $0 \xrightarrow{28} 1$



(a) (b) (c)


1 0 10 5
2 2 12 3
3 1 14 6
4 1 16 2
5 3 18 6
6 3 22 4
7 4 24 6
8 4 25 5
9 0 28 1
0 1 5 6 2 4 3 10 12
0 1 5 6 2 4 3 10 14 12
0 1 5 6 2 4 3 10 14 16 12
(d)
(e)
(f)

1 $0 \xrightarrow{10} 5$

2 $2 \xrightarrow{12} 3$

3 $1 \xrightarrow{14} 6$

4 $1 \xrightarrow{16} 2$

5 $3 \xrightarrow{18} 6$

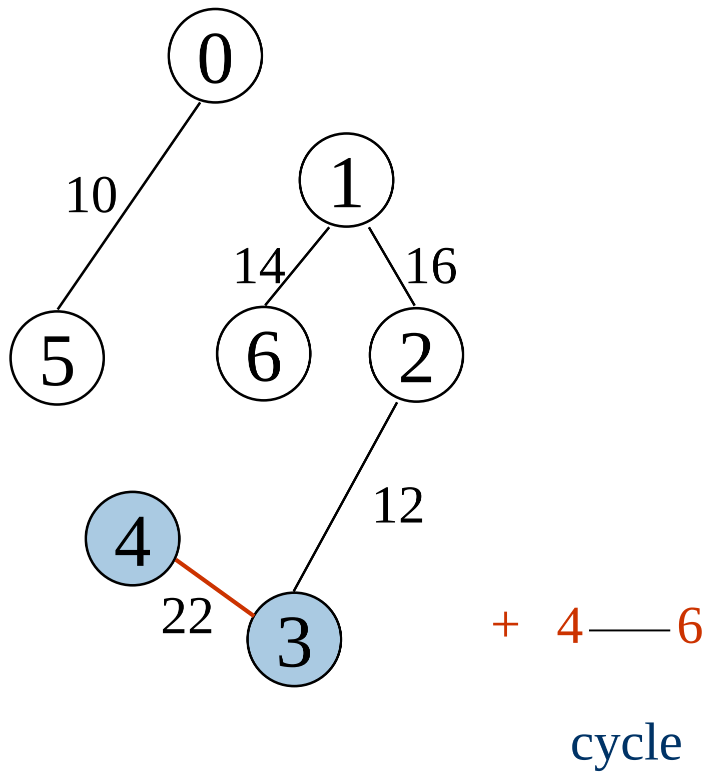
6 $3 \xrightarrow{22} 4$

7 $4 \xrightarrow{24} 6$

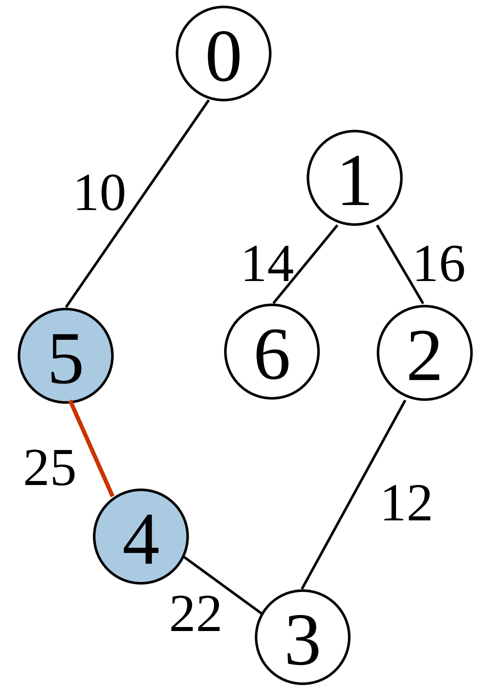
8 $4 \xrightarrow{25} 5$

9 $0 \xrightarrow{28} 1$

cost = 10 +25+22+12+16+14



(g)



(h)

Kruskal's Algorithm

目標：取出n-1條edges

```
T= {};
while (T contains less than n-1 edges
        && E is not empty) {
choose a least cost edge (v,w) from E;
delete (v,w) from E;
if ((v,w) does not create a cycle in T)
    add (v,w) to T
 else discard (v,w);
}
if (T contains fewer than n-1 edges)
  printf("No spanning tree\n");
```

min heap construction time O(e)

choose and delete O(log e)

find find & union O(log e)

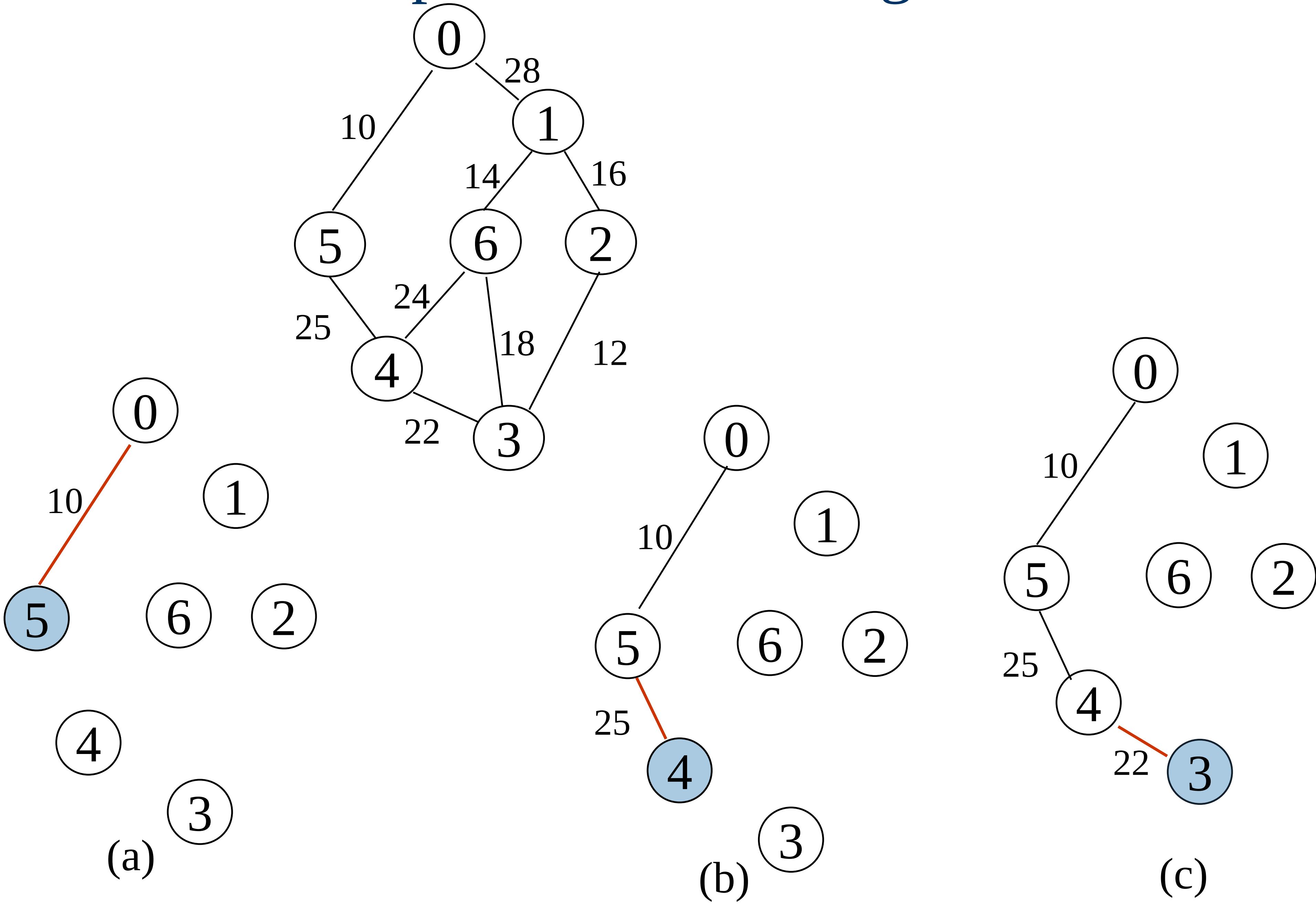
O(e log e)

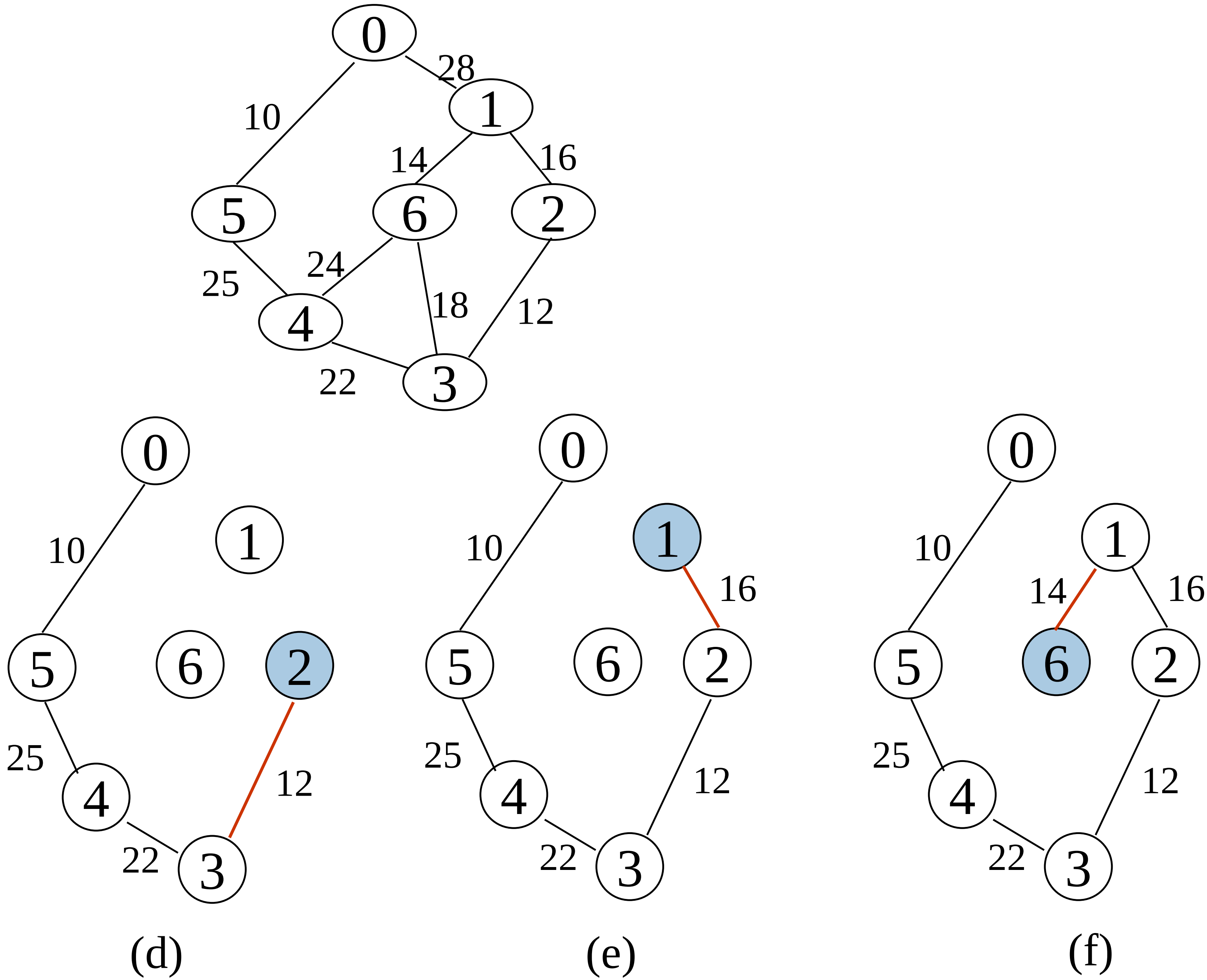
Prim's Algorithm

(tree all the time vs. forest)

```
T={};
TV={0};
while (T contains fewer than n-1 edges)
{
  let (u,v) be a least cost edge such
    that u ∈ TV and v ∉ TV
  if (there is no such edge ) break;
  add v to TV;
  add (u,v) to T;
}
if (T contains fewer than n-1 edges)
  printf(“No spanning tree\n”);
```

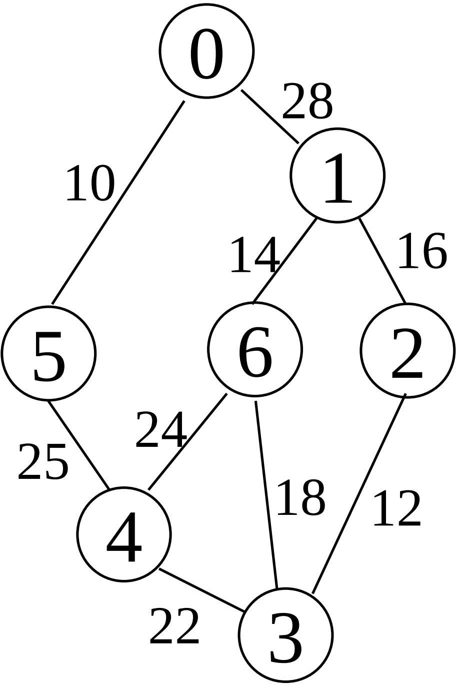
Examples for Prim's Algorithm



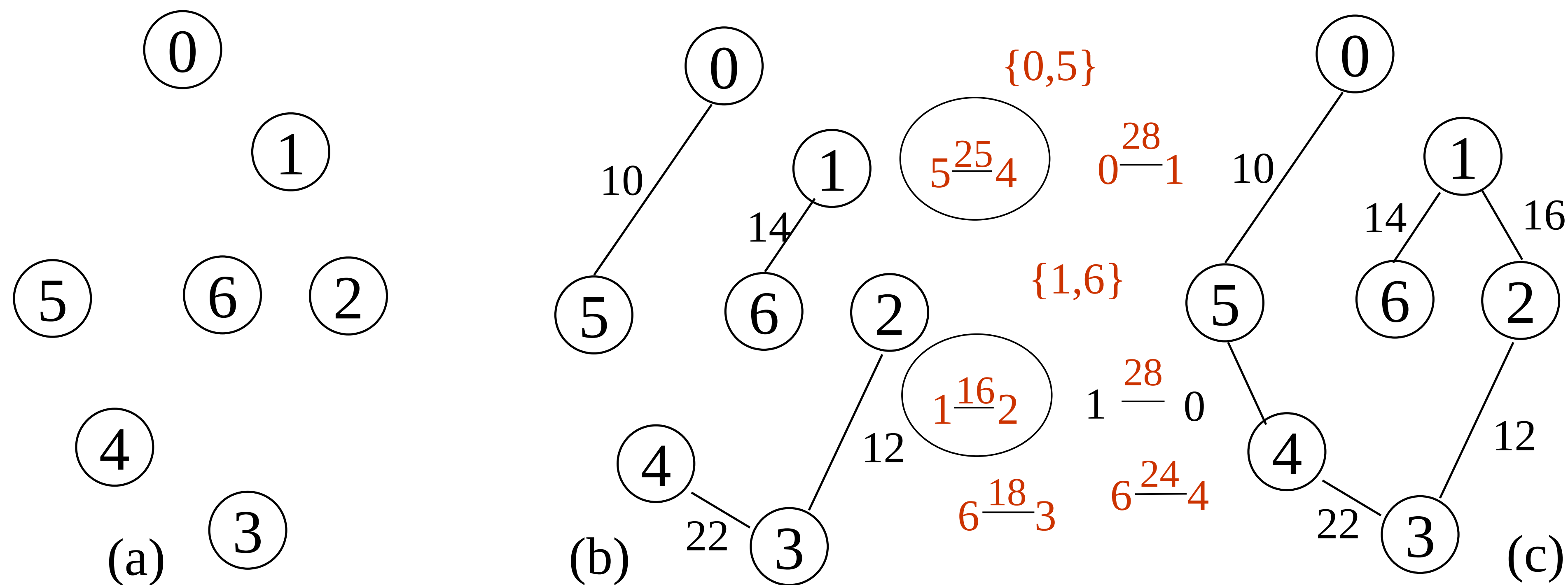


0
28
1
10
14
16
5
6
2
25
24
18
12
4
22
3
(d)
(e)
(f)

Sollin's Algorithm

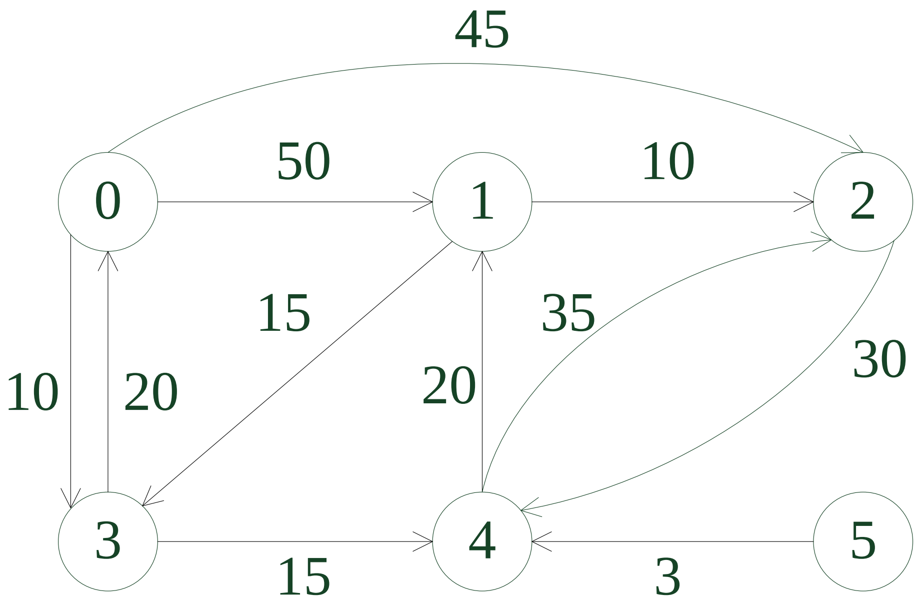


vertex	edge
0	0 -- 10 --> 5, 0 -- 28 --> 1
1	1 -- 14 --> 6, 1-- 16 --> 2, 1 -- 28 --> 0
2	2 -- 12 --> 3, 2 -- 16 --> 1
3	3 -- 12 --> 2, 3 -- 18 --> 6, 3 -- 22 --> 4
4	4 -- 22 --> 3, 4 -- 24 --> 6, 4 -- 25 --> 5
5	5 -- 10 --> 0, 5 -- 25 --> 4
6	6 -- 14 --> 1, 6 -- 18 --> 3, 6 -- 24 --> 4



Single Source to All Destinations

Determine the shortest paths from v0 to all the remaining vertices.



(a) 圖

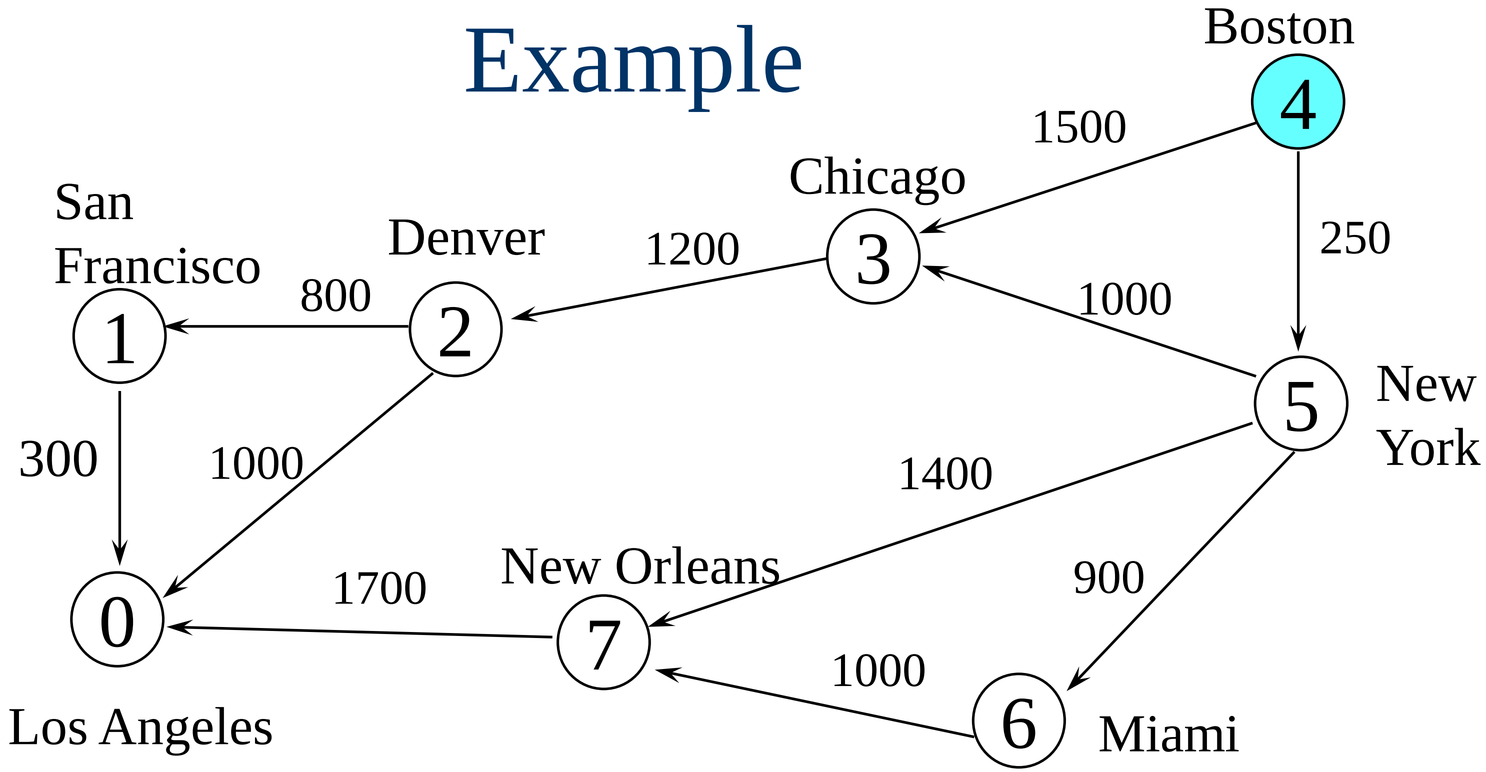
Dijkstra's algorithm

	路徑	長度
1)	0, 3	10
2)	0, 3, 4	25
3)	0, 3, 4, 1	45
4)	0, 2	45

(b) 從 0 出發的最短路徑

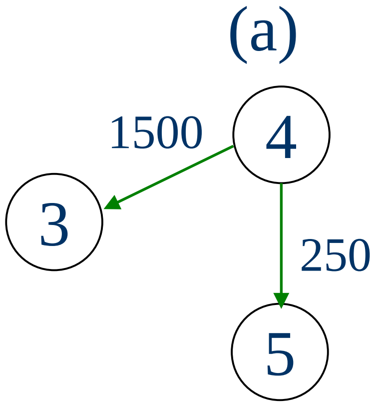
***Figure 6.26:** Graph and shortest paths from v_0*

Example

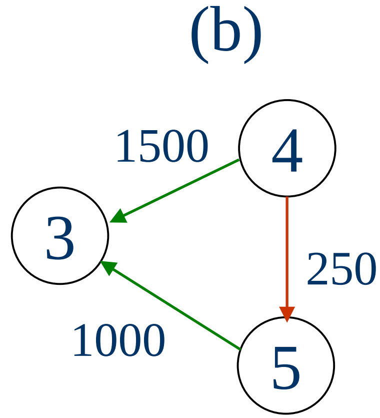


	0	1	2	3	4	5	6	7
0	0							
1	300	0						
2	1000	800	0					
3			1200	0				
4				1500	0	250		
5				1000		0	900	1400
6							0	1000
7	1700							0

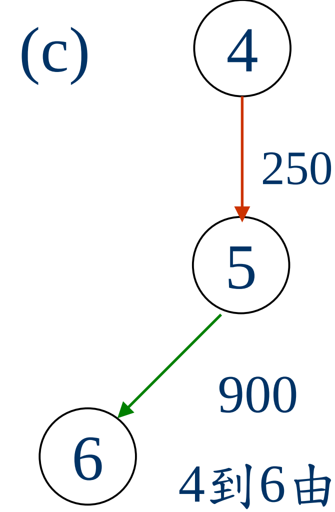
Cost adjacency matrix



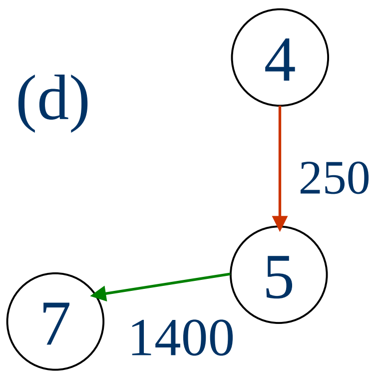
選5



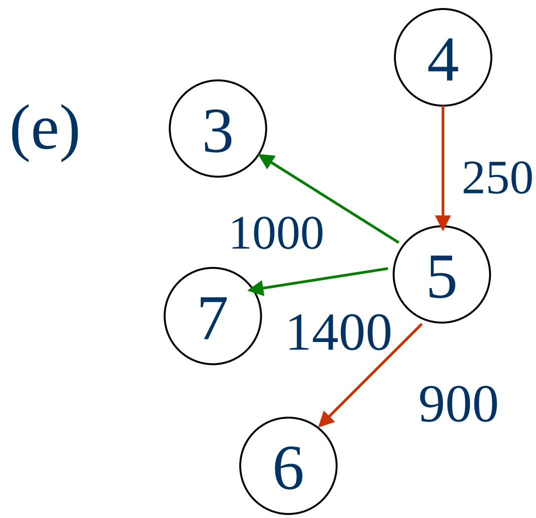
4到3由1500改成1250



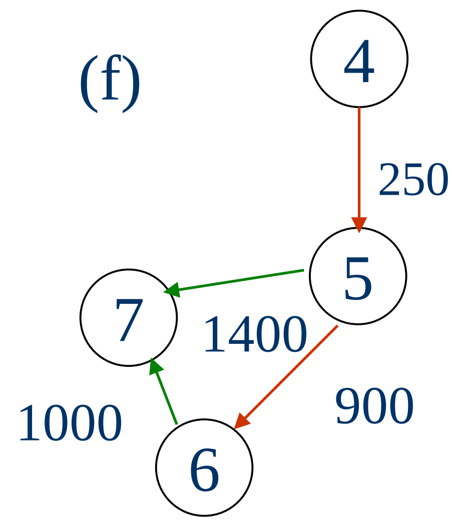
4到6由∞改成1150



4到7由∞改成1650

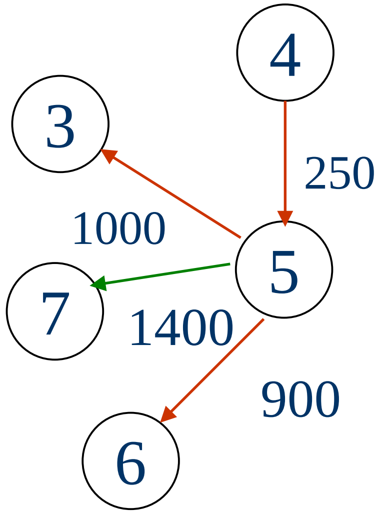


選6



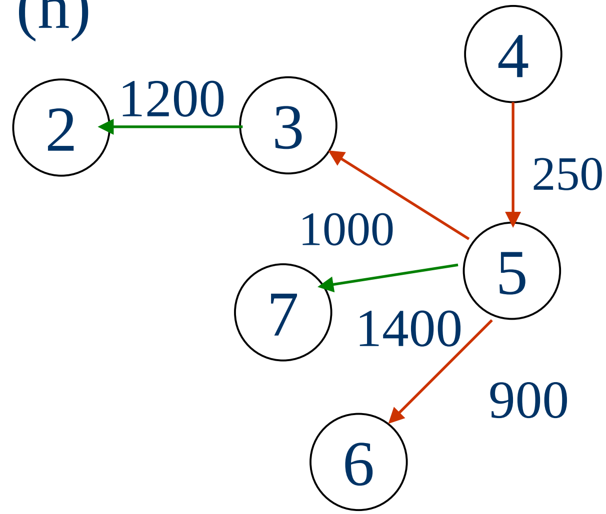
4-5-6-7比4-5-7長

(g)



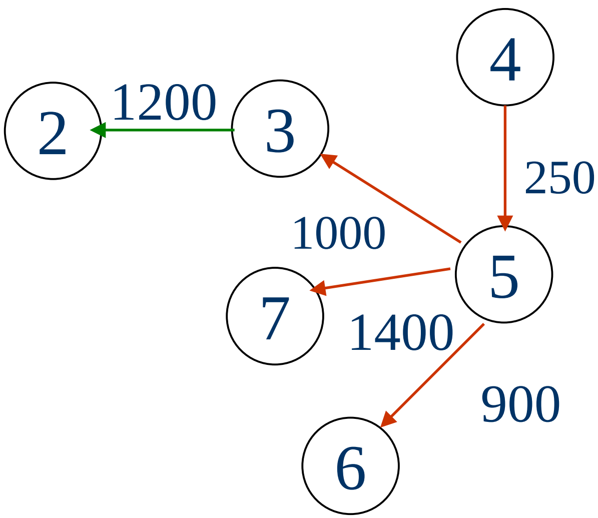
選3

(h)



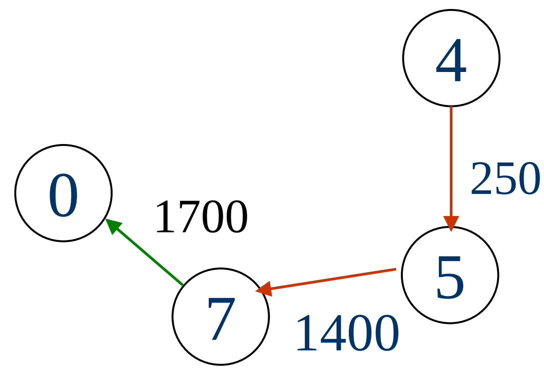
4到2由∞改成2450

(i)



選7

(j)



4到0由∞改成3350

Example for the Shortest Path

(Continued)

Iteration	S	Vertex Selected	LA [0]	SF [1]	DEN [2]	CHI [3]	BO [4]	NY [5]	MIA [6]	NO
Initial	--	----	+∞	+∞	+∞	1500	0	250	+∞	+∞
1	{4} (a)	5	+∞	+∞	+∞ (b)	1250	0	250 (c)	1150 (d)	1650
2	{4,5} (e)	6	+∞	+∞	+∞	1250	0	250	1150 (f)	1650
3	{4,5,6} (g)	3	+∞	+∞ (h)	2450	1250	0	250	1150	1650
4	{4,5,6,3} (i)	7	(j) 3350	+∞	2450	1250	0	250	1150	1650
5	{4,5,6,3,7}	2	3350	3250	2450	1250	0	250	1150	1650
6	{4,5,6,3,7,2}	1	3350	3250	2450	1250	0	250	1150	1650
7	{4,5,6,3,7,2,1}									

Single Source to All Destinations

```
void shortestpath(int v, int
  cost[][MAX_ERXTICES], int distance[], int n,
  short int found[])
{
  int i, u, w;
  for (i=0; i<n; i++) {
    found[i] = FALSE;
    distance[i] = cost[v][i];
  }
  found[v] = TRUE;
  distance[v] = 0;
```

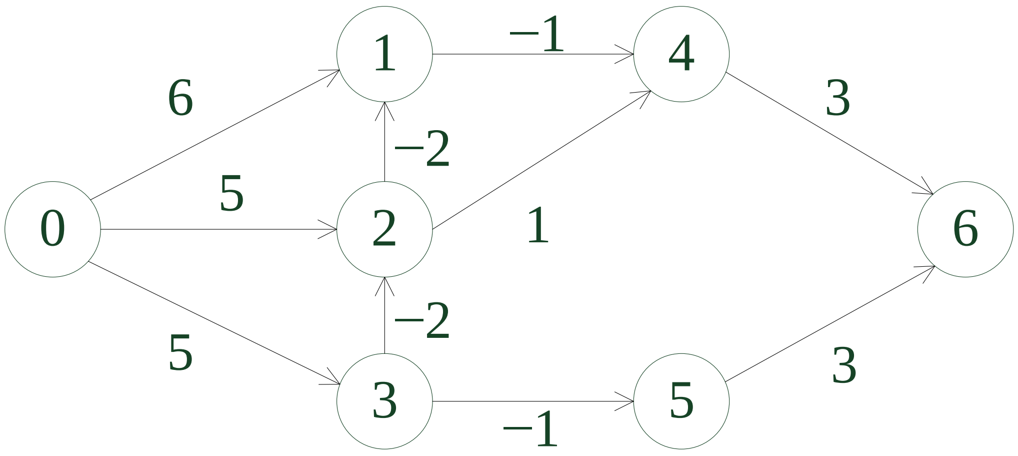
O(n)

```
for (i=0; i<n-2; i++) {determine n-1 paths from v
  u = choose(distance, n, found);
  found[u] = TRUE;
  for (w=0; w<n; w++)
    if (!found[w])    與u相連的端點w
      if (distance[u]+cost[u][w]<distance[w])
        distance[w] = distance[u]+cost[u][w];
  }
}
```

$O(n^2)$

```
int choose(int distance[], int n, short int found[])
{
  /* 找出還沒確認最短距離的點 */
  int i, min, minpos;
  min = INT_MAX;
  minpos = -1;
  for (i = 0; i < n; i++) {
   if(distance[i] < min && !found[i]){
     min = distance[i];
     minpos = i;
   }
  return minpos;
}
```

Shortest paths with negative edge lengths



(a) 有向圖

k	$dist^k[7]$ 0	1	2	3	4	5	6
1	0	6	5	5	∞	∞	∞
2	0	3	3	5	5	4	∞
3	0	1	3	5	2	4	7
4	0	1	3	5	0	4	5
5	0	1	3	5	0	4	3
6	0	1	3	5	0	4	3

(b) $dist^k$

Bellman and Ford algorithm to compute shortest paths

```
void BellmanFord(int n, int v)
 {
   /* 計算單一起點/所有終點的最短路徑，其中邊長允許是負值 */
   for (int i = 0; i < n; i++)
        dist[i] = length[v][i];
          /* 對dist做初始化 */

   for (int k = 2; k <= n-1; k++)
     for (每個u滿足u!=v 且u至少有一個進到它的邊)
       for(每個圖上的邊<i,u>)
        if(dist[u] > dist[i] + length[i][u])
           dist[u] = dist[i] + length[i][u];
 }
```

All Pairs Shortest Paths

- Find the shortest paths between all pairs of vertices.
- Solution 1
  - Apply shortest path *n* times with each vertex as source.

    $O(n^3)$
- Solution 2
  - Represent the graph G by its cost adjacency matrix with cost[i][j]
  - If the edge <i,j> is not in G, the cost[i][j] is set to some sufficiently large number
  - A[i][j] is the cost of the shortest path form i to j, using only those intermediate vertices with an index <= k

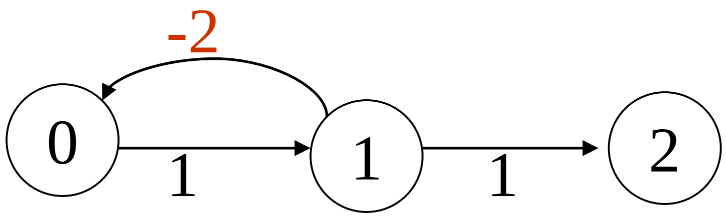
All Pairs Shortest Paths (*Continued*)

- The cost of the shortest path from i to j is $A^{n-1}[i][j]$, as no vertex in G has an index greater than n-1
- $A^{-1}[i][j]=cost[i][j]$
- Calculate the $A^0, A^1, A^2, \ldots, A^{n-1}$ from A^{-1} iteratively
- $A^k[i][j]=\min\{A^{k-1}[i][j], A^{k-1}[i][k]+A^{k-1}[k][j]\}, k>=0$

Algorithm for All Pairs Shortest Paths

```
void allcosts(int cost[][MAX_VERTICES],
         int distance[][MAX_VERTICES], int n)
{
  int i, j, k;
  for (i=0; i<n; i++)
    for (j=0; j<n; j++)
         distance[i][j] = cost[i][j];
  for (k=0; k<n; k++)
    for (i=0; i<n; i++)
      for (j=0; j<n; j++)
        if (distance[i][k]+distance[k][j]
            < distance[i][j])
           distance[i][j]=
               distance[i][k]+distance[k][j];
}
```

Graph with Negative Cycle



(a) Directed graph

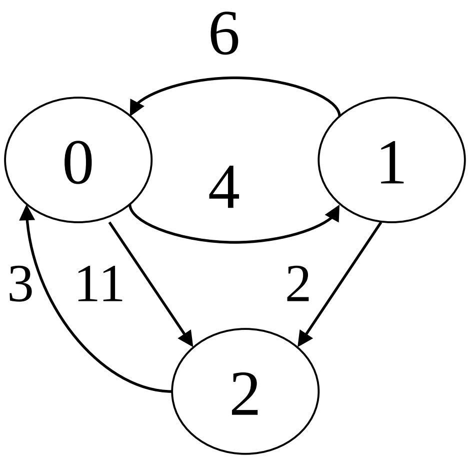
$$\begin{bmatrix} 0 & 1 & \infty \\ -2 & 0 & 1 \\ \infty & \infty & 0 \end{bmatrix}$$

(b) A^{-1}

0, 1, 0, 1, 0, 1, …, 0, 1, 2

The length of the shortest path from vertex 0 to vertex 2 is $-\infty$.

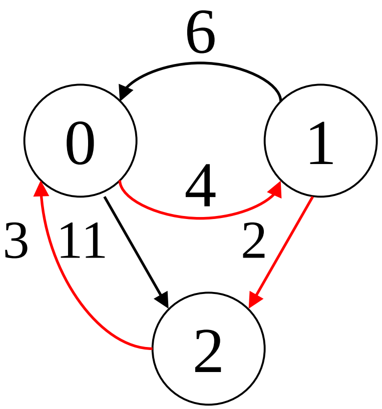
* **Figure 6.33:** Directed graph and its cost matrix



(a)Directed graph G

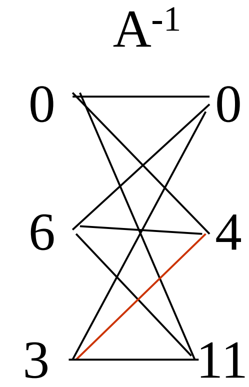
	0	1	2
0	0	4	11
1	6	0	2
2	3	∞	0

(b)Cost adjacency matrix for G

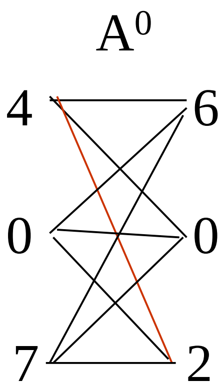


A^{-1}	0	1	2
0	0	4	11
1	6	0	2
2	3	∞	0

A^{0}	0	1	2
0	0	4	11
1	6	0	2
2	3	7	0



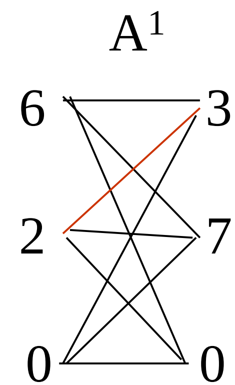
V0 加入



V1 加入

A^{1}	0	1	2
0	0	4	6
1	6	0	2
2	3	7	0

A^{2}	0	1	2
0	0	4	6
1	5	0	2
2	3	7	0



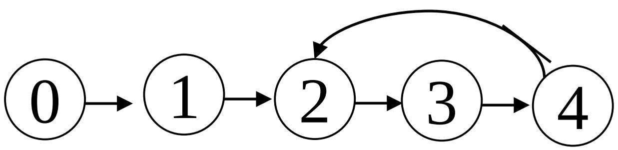
V2 加入

Transitive Closure

Goal: given a graph with unweighted edges, determine if there is a path from i to j for all i and j.

(1) Require positive path (> 0) lengths. transitive closure matrix

(2) Require nonnegative path (≥0) lengths. reflexive transitive closure matrix



(a) Digraph G

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

(b) Adjacency matrix A for G

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

cycle

(c) transitive closure matrix A^+

There is a path of length > 0

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

reflexive

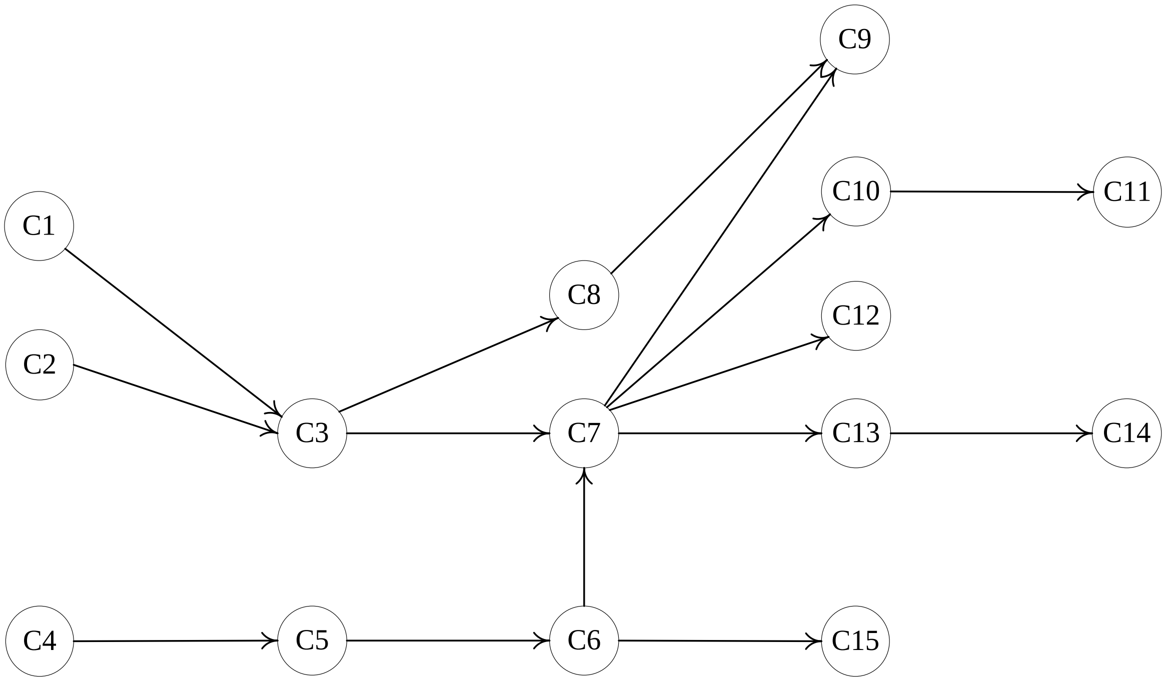
(d) reflexive transitive closure matrix A^*

There is a path of length ≥0

Activity on Vertex (AOV) Network

- Definition: A directed graph in which the vertices represent tasks or activities and the edges represent precedence relations between tasks.
- Predecessor (successor): vertex *i* is a predecessor of vertex *j* iff there is a directed path from i to j.
  - j is a successor of i.
- Partial order: a precedence relation which is both transitive ($\forall$i, j, k, i•j & j•k => i•k) and irreflexive ($\forall$x ¬x•x).
- Acylic graph: a directed graph with no directed cycles

*Figure 6.37: An AOV network



課程編號	課程名稱	先修課程
C1	程式I	無
C2	離散數學	無
C3	資料結構	C1, C2
C4	微積分I	無
C5	微積分II	C4
C6	線性代數	C5
C7	演算法分析	C3, C6
C8	組合語言	C3
C9	作業系統	C7, C8
C10	程式語言	C7
C11	編譯器設計	C10
C12	人工智慧	C7
C13	計算機理論	C7
C14	平行演算法	C13
C15	數值分析	C5

Topological order:
linear ordering of vertices of a graph
∀i, j if *i* is a predecessor of j, then i precedes *j* in the linear ordering

C1, C2, C4, C5, C3, C6, C8, C7, C10, C13, C12, C14, C15, C11, C9

C4, C5, C2, C1, C6, C3, C8, C15, C7, C9, C10, C11, C13, C12, C14

***Program 6.13:** Topological sort

```
for (i = 0; i <n; i++) {
      if every vertex has a predecessor {
         fprintf(stderr, "Network has a cycle. \n " );
         exit(1);
       }
       pick a vertex v that has no predecessors;
       output v;
       delete v and all edges leading out of v
       from the network;
}
```

*Figure 6.38: Simulation of Program 6.13 on an AOV network

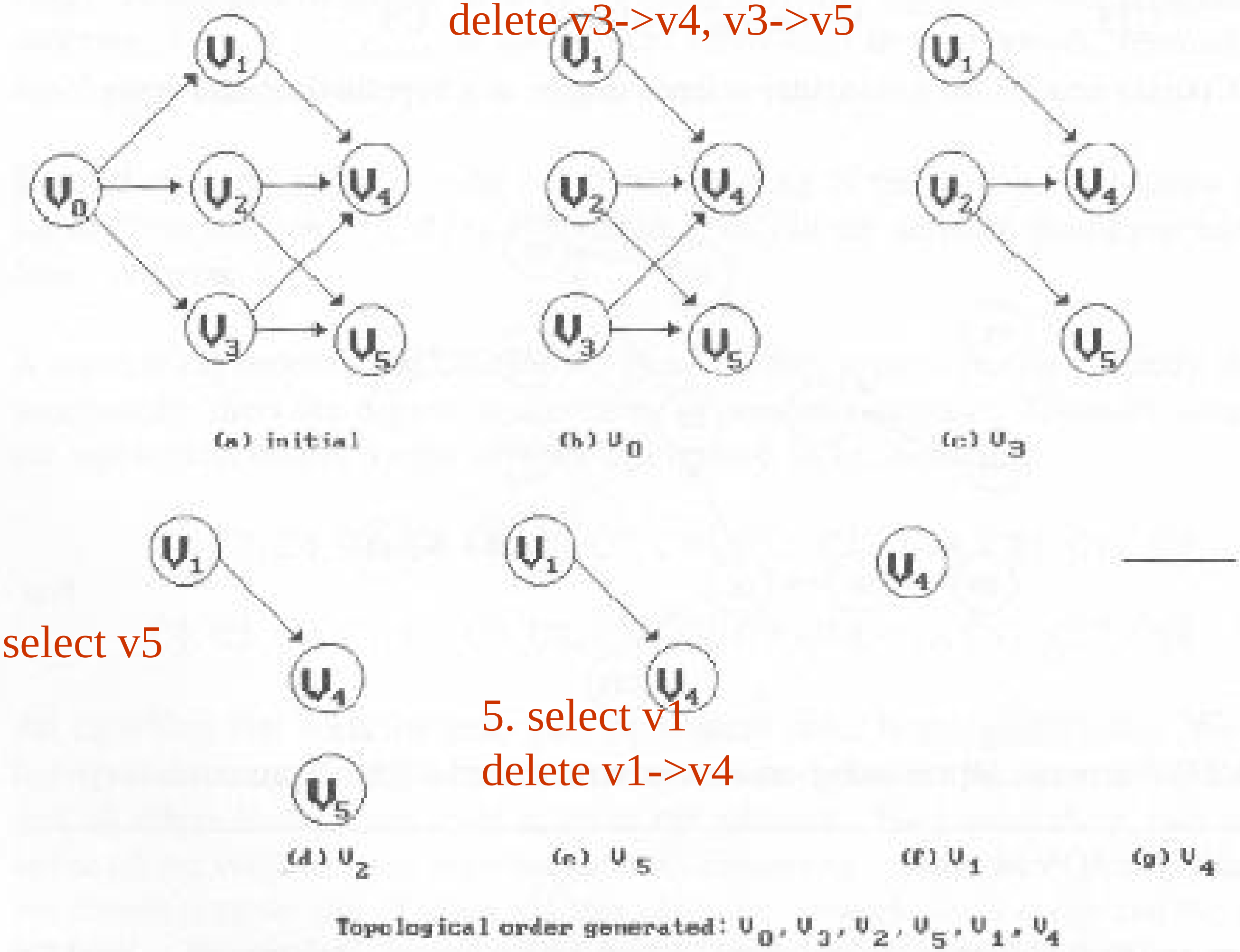
1. v0 no predecessor
delete v0->v1, v0->v2, v0->v3

2. v1, v2, v3 no predecessor
select v3
delete v3->v4, v3->v5

3. select v2
delete v2->v4, v2->v5

4. select v5

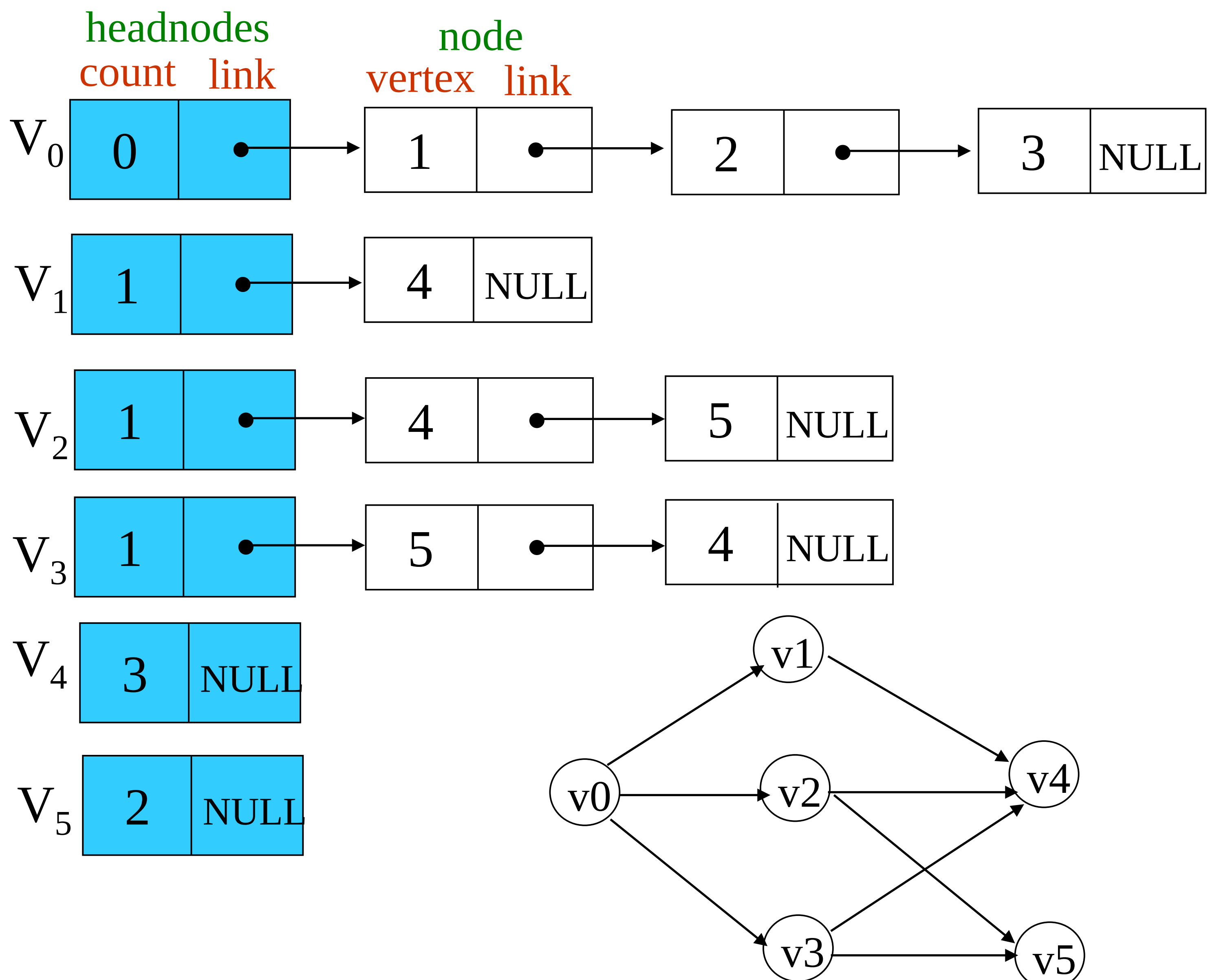
5. select v1
delete v1->v4



Issues in Data Structure Consideration

- Decide whether a vertex has any predecessors.
  - –Each vertex has a count.
- Decide a vertex together with all its incident edges.
  - –Adjacency list

*Figure 6.39: Internal representation used by topological sorting algorithm



```
typedef struct node *node_pointer;
typedef struct node {
          int vertex;
          node_pointer link;
          };
typedef struct {
          int count;
          node_pointer link;
          } hdnodes;
hdnodes graph[MAX_VERTICES];
```

***Program 6.14:** Topological sort

```
void topsort (hdnodes graph [] , int n)
{
  int i, j, k, top;
  node_pointer ptr;
  /* create a stack of vertices with no predecessors */
 top = -1;
 for (i = 0; i < n; i++)
     if (!graph[i].count) {no predecessors, stack is linked through count field
        graph[i].count = top;
        top = i;
 }
for (i = 0; i < n; i++)
   if (top == -1) {
      fprintf(stderr, "\n Network has a cycle. Sort terminated. \n");
      exit(1);
}
```

O(n)

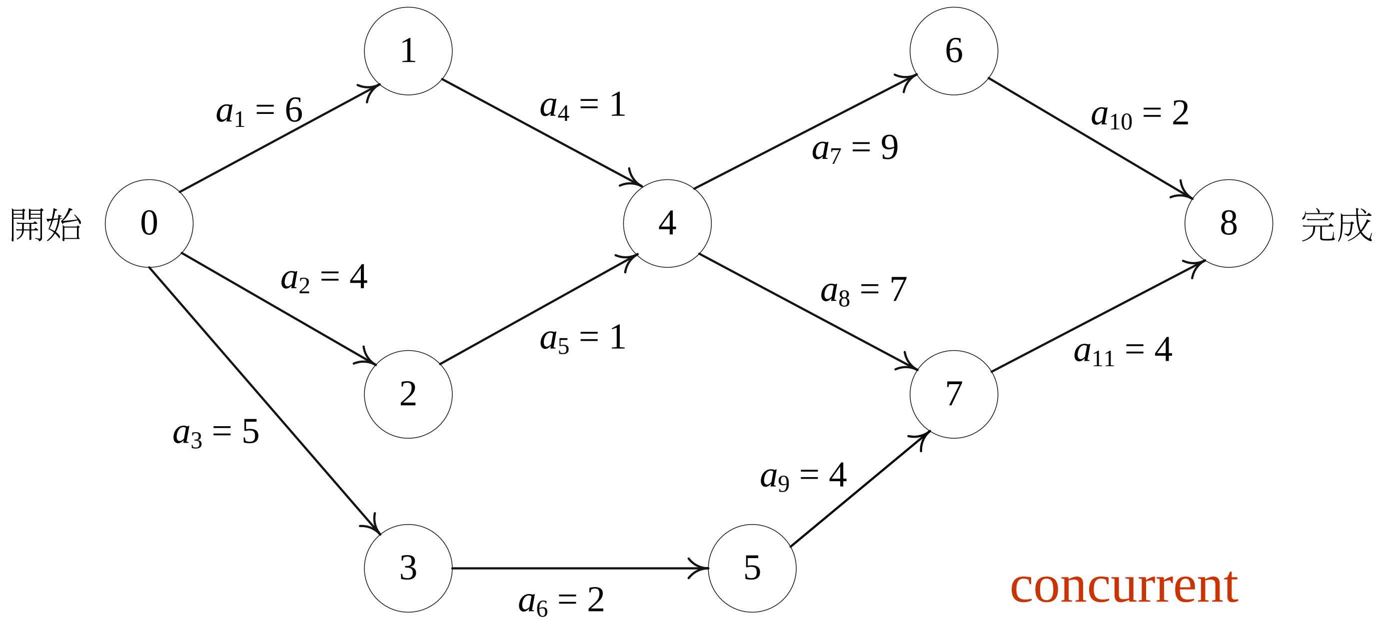
```
}
    else {
        j = top; /* unstack a vertex */
        top = graph[top].count;
        printf("v%d, ", j);
        for (ptr = graph [j].link; ptr ; ptr = ptr ->link ){
        /* decrease the count of the successor vertices of j */
            k = ptr ->vertex;
            graph[k].count --;
            if (!graph[k].count) {
            /* add vertex k to the stack*/
                graph[k].count = top;
                top = k;
            }
        }
    }
}
```

O(e)

O(e+n)

Activity on Edge (AOE) Networks

- Directed edge
  - tasks or activities to be performed
- Vertex
  - events which signal the completion of certain activities
- Number
  - time required to perform the activity



(Fig. 6.40)

事件	解釋
0	計劃開始
1	活動$\mathbf{a_1}$完成
4	活動$\mathbf{a_4}$和$\mathbf{a_5}$完成
7	活動$\mathbf{a_8}$和$\mathbf{a_9}$完成
8	計畫完成

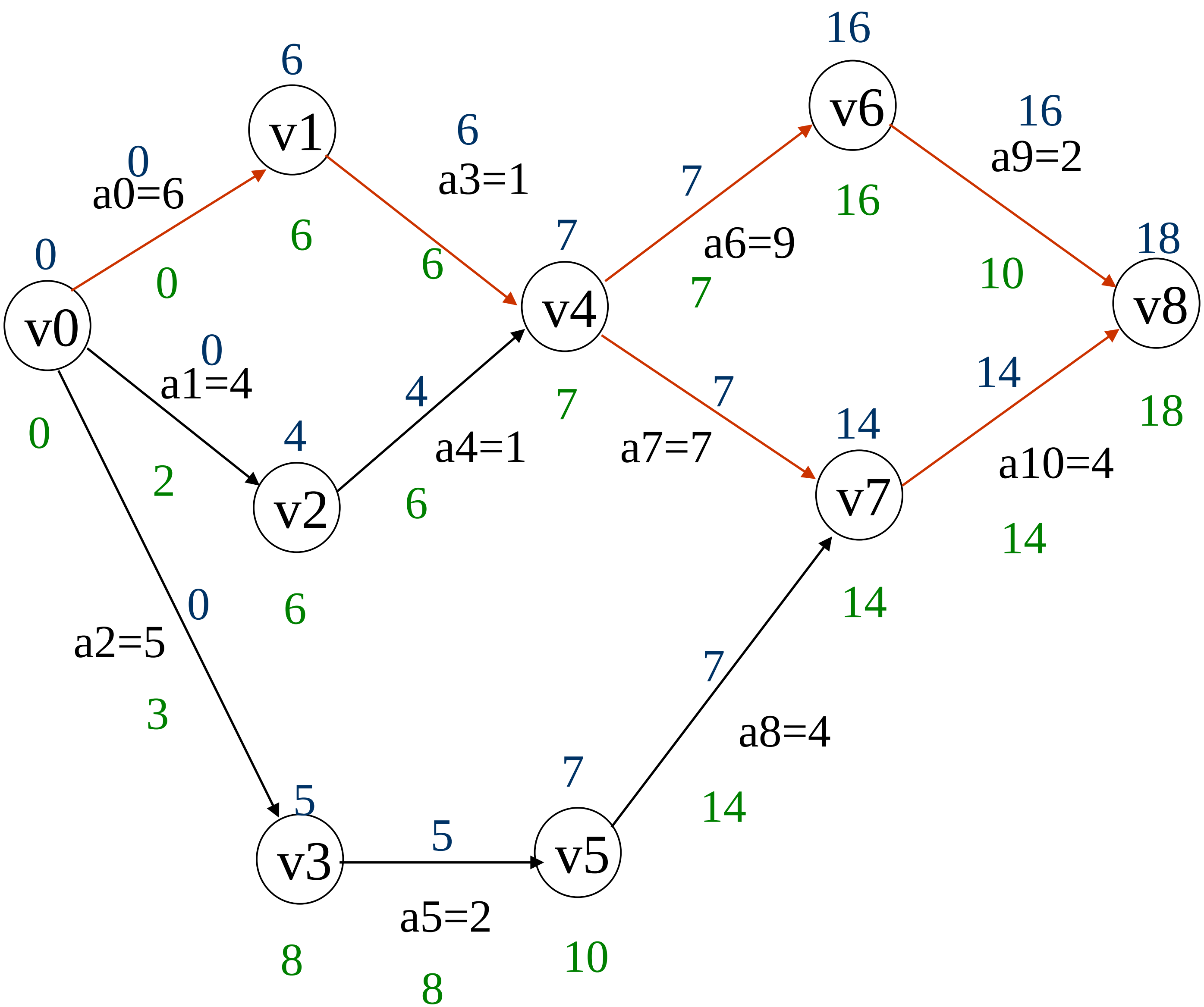
Application of AOE Network

- Evaluate performance
  - minimum amount of time
  - activity whose duration time should be shortened
  - …
- Critical path
  - a path that has the longest length
  - minimum time required to complete the project
  - v0, v1, v4, v7, v8 or v0, v1, v4, v6, v8

AOE

- Earliest time that vi can occur
  - the length of the longest path from v0 to vi
  - the earliest start time for all activities leaving vi
  - early(7) = early(8) = 7
- Latest time of activity
  - the latest time the activity may start without increasing the project duration
  - late(6) = 8, late(8) = 7
- Critical activity
  - an activity for which early(i)=late(i)
  - early(7)=late(7)=14
- late(i)-early(i)
  - measure of how critical an activity is
  - late(5)-early(5)=10-7=3

earliest, early, latest, late



Determine Critical Paths

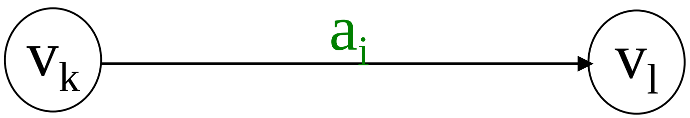
- Delete all noncritical activities
- Generate all the paths from the start to finish vertex.

Calculation of Earliest Times

- earliest[j]
  - the earliest event occurrence time

$$earliest[0]=0$$
$$earliest[j]=\max_{i \in p(j)}\{earliest[i]+\text{duration of } \langle i,j\rangle\}$$

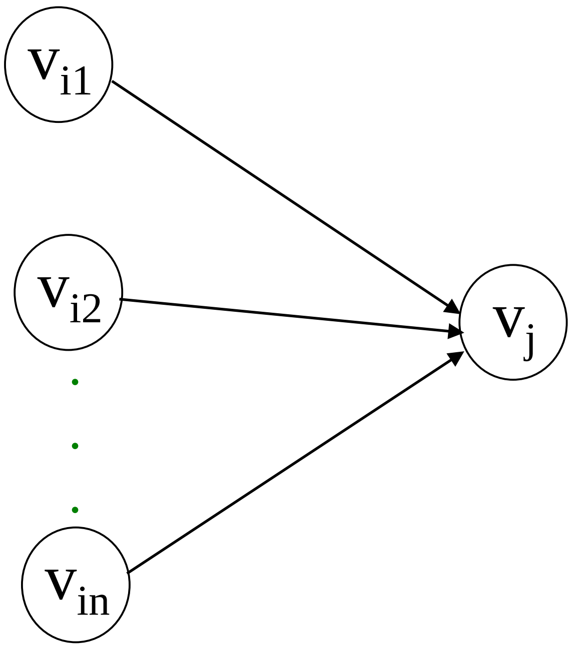
- latest[j]
  - the latest event occurrence time



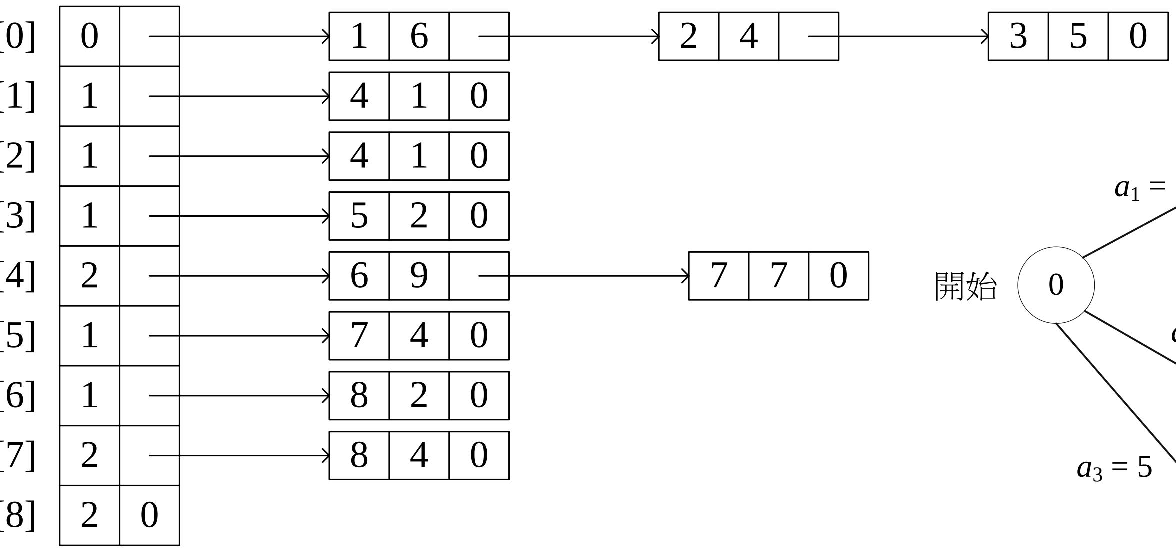
early(i)=earliest(k)

late(i)=latest(l)-duration of a_i

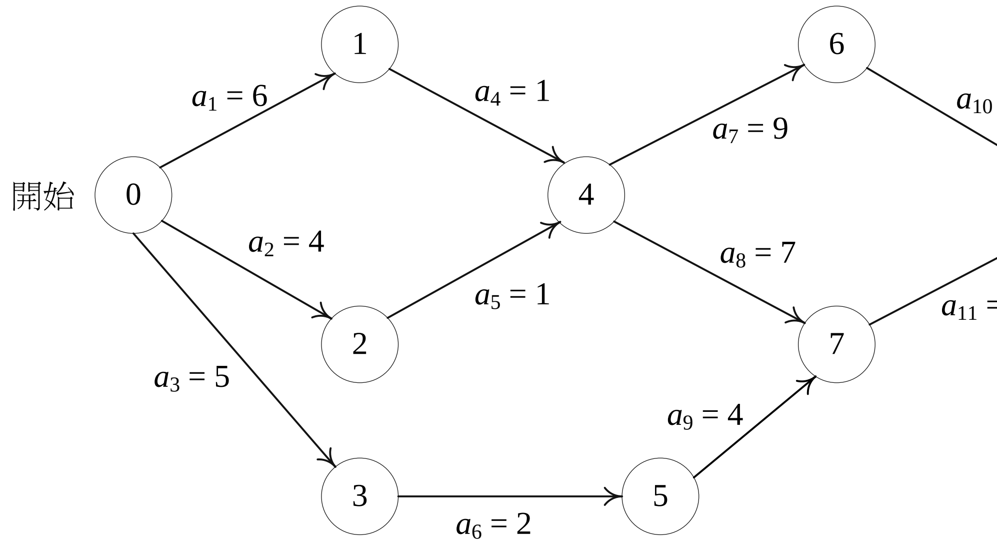
forward stage



if (earliest[k] < earliest[j]+ptr->duration)
earliest[k]=earliest[j]+ptr->duration



(a) 圖 6.40(a) 的相鄰串列



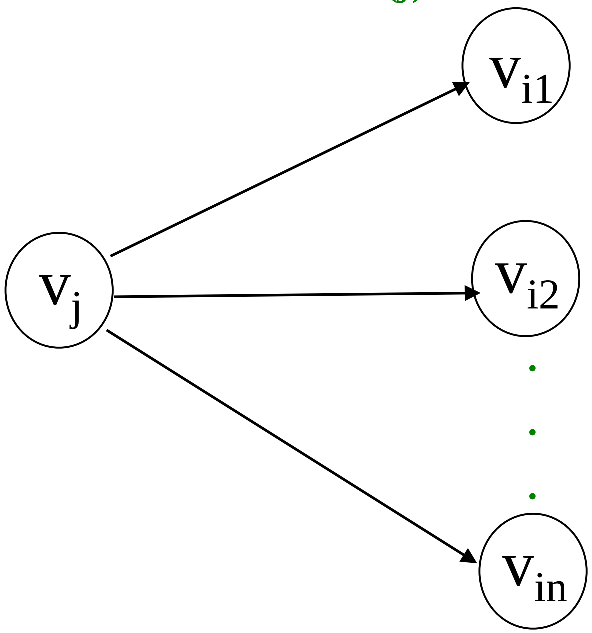
ee	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	堆疊
起始	0	0	0	0	0	0	0	0	0	[0]
輸出 0	0	6	4	5	0	0	0	0	0	[3, 2, 1]
輸出 3	0	6	4	5	0	7	0	0	0	[5, 2, 1]
輸出 5	0	6	4	5	0	7	0	11	0	[2, 1]
輸出 2	0	6	4	5	5	7	0	11	0	[1]
輸出 1	0	6	4	5	7	7	0	11	0	[4]
輸出 4	0	6	4	5	7	7	16	14	0	[7, 6]
輸出 7	0	6	4	5	7	7	16	14	18	[6]
輸出6	0	6	4	5	7	7	16	14	18	[8]
輸出 8										

(b) *ee* 的計算

Calculation of Latest Times

- latest[j]
  - the latest event occurrence time

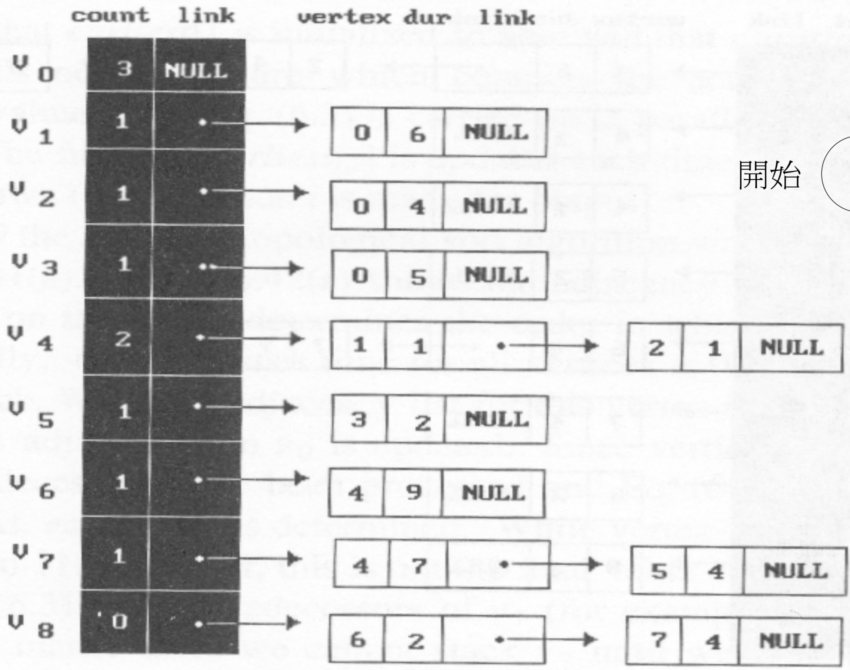
$$\text{latest}[j]=\min_{i\in s(j)}\{\text{latest}[i]\text{-duration of} <j,i>\}$$



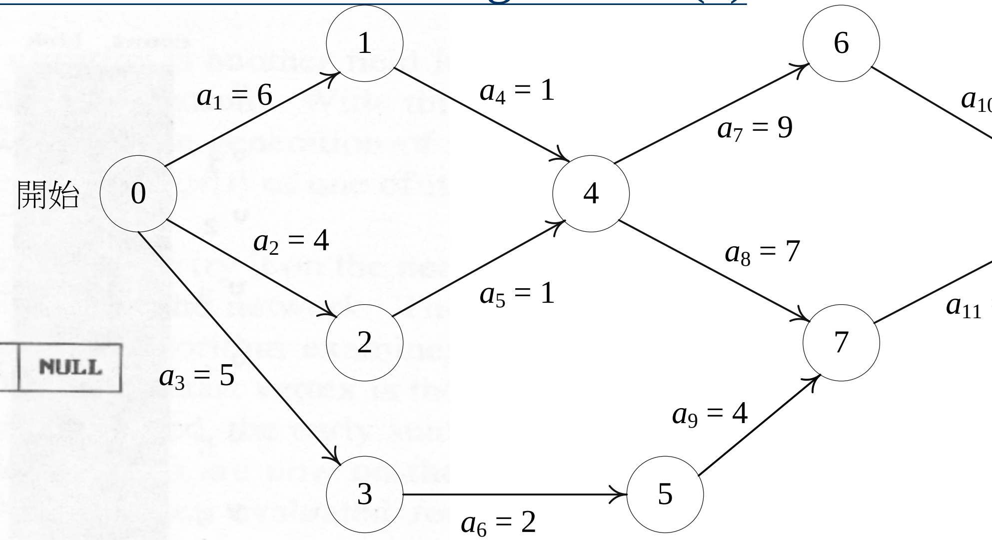
backward stage

if (latest[k] > latest[j]-ptr->duration)
  latest[k]=latest[j]-ptr->duration

*Figure 6.43: Computing latest for AOE network of Figure 6.41(a)



(a) Inverted adjacency lists for AOE network of Figure 6.41(a)



Latest	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	Stack
initial	18	18	18	18	18	18	18	18	18	[8]
output v_8	18	18	18	18	18	18	16	14	18	[7, 6]
output v_7	18	18	18	18	7	10	16	14	18	[5, 6]
output v_5	18	18	18	18	7	10	16	14	18	[3, 6]
output v_3	3	18	18	8	7	10	16	14	18	[6]
output v_6	3	18	18	8	7	10	16	14	18	[4]
output v_4	3	6	6	8	7	10	16	14	18	[2, 1]
output v_2	2	6	6	8	7	10	16	14	18	[1]
output v_1	0	6	6	8	7	10	16	14	18	[0]

(b) Computation of *latest*

***Figure 6.43(continued):**Computing latest of AOE network of Figure 6.41(a)

latest[8]=earliest[8]=18
latest[6]=min{le[8] - 2}=16
latest[7]=min{le[8] - 4}=14
latest[4]=min{le[6] - 9; le[7] -7}= 7
latest[1]=min{le[4] - 1}=6
latest[2]=min{le[4] - 1}=6
latest[5]=min{le[7] - 4}=10
latest[3]=min{le[5] - 2}=8
latest[0]=min{le[1] - 6; le[2]- 4; le[3] -5}=0

(c) Computation of latest from Equation (6.3) using a reverse topological order

*Figure 6.42: Early, late and critical values

Activity	Early	Late	Late-Early	Critical
a_1	0	0	0	Yes
a_2	0	2	2	No
a_3	0	3	3	No
a_4	6	6	0	Yes
a_5	4	6	2	No
a_6	5	8	3	No
a_7	7	7	0	Yes
a_8	7	7	0	Yes
a_9	7	10	3	No
a_{10}	16	16	0	Yes
a_{11}	14	14	0	Yes

$l - e = 0$

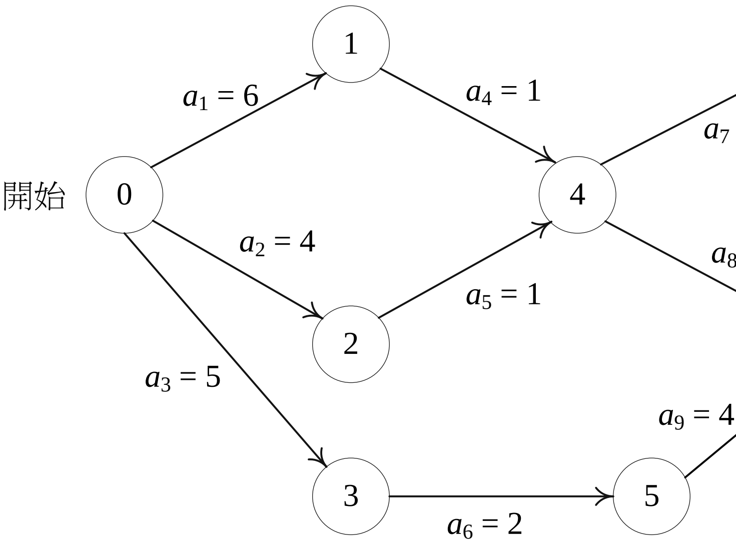


***Figure 6.43:** Graph with noncritical activities deleted

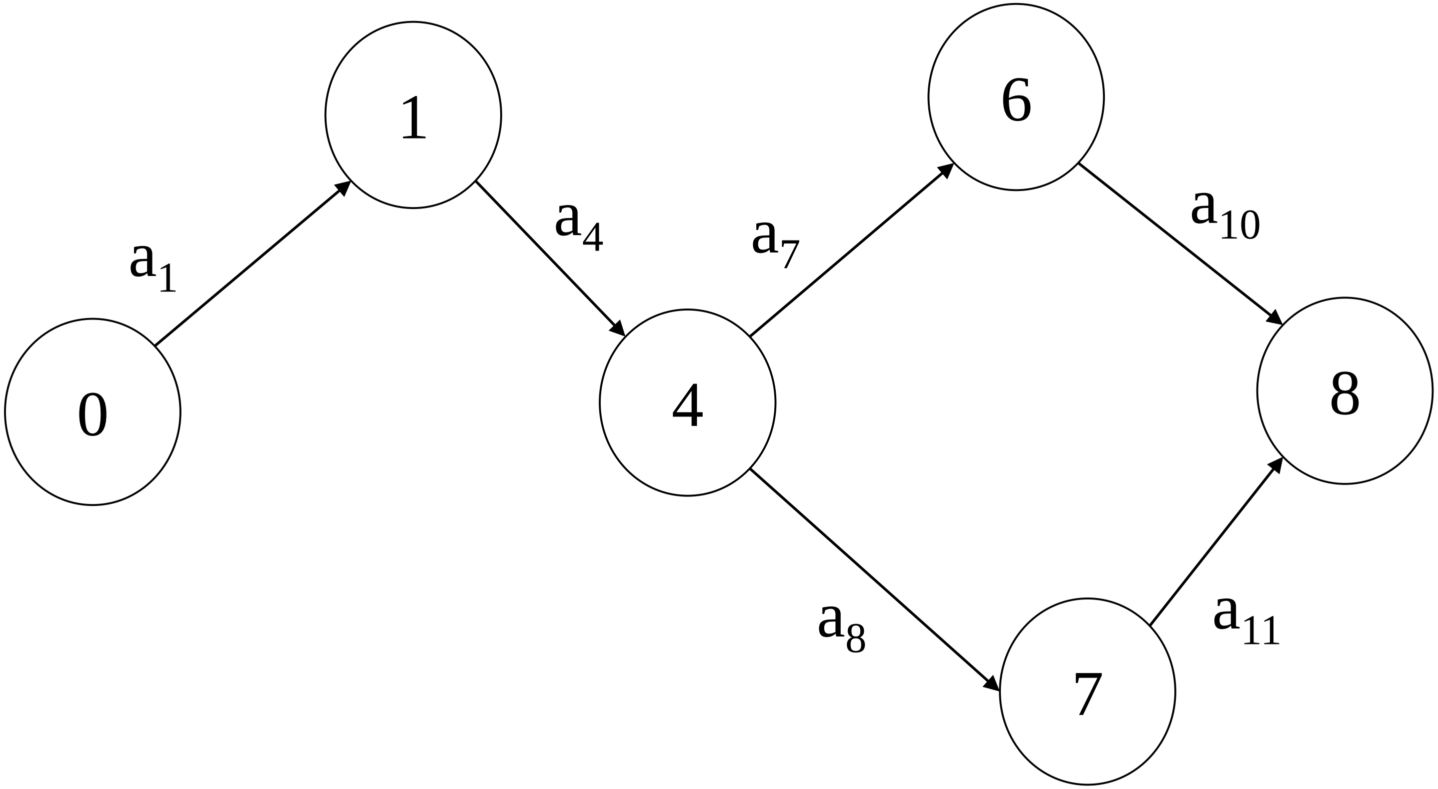


***Figure 6.45:** AOE network with unreachable activities